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ECONOMETRICA

VOLUME 6 OCTOBER, 1938 NUMBER 4

ÉTUDE PARTICULIÈRE D'UNE LOI DE DEMANDE: LE TRAFIC POSTAL EN FRANCE DE 1873 A 1936

By RENÉ ROY

INTRODUCTION

IL Y A EXACTEMENT un siècle, Augustin Cournot publiait ses *Recherches sur les Principes Mathématiques de la Théorie des Richesses*"; la célébration de ce centenaire a donné lieu à plusieurs manifestations, tant à l'étranger qu'en France, et nous avons eu l'occasion de participer à certaines de ces démonstrations.

Estimant toutefois que la meilleure façon d'honorer nos maîtres est de nous engager plus avant dans la voie qu'ils ont tracée, nous avons pensé qu'il ne serait pas dénué d'intérêt de présenter aux lecteurs d'ECONOMETRICA une étude qui, tout en faisant appel à des procédés statistiques auxquels Cournot ne pouvait songer, s'inspirent directement de ses conceptions en matière de débit. Cette étude, que son auteur a effectuée d'accord avec nous, est due à M. Morice, Professeur de mathématiques; elle porte sur le trafic postal en France de 1873 à 1936.

Nous avons toujours pensé en effet que les services monopolisés convenaient particulièrement aux recherches économétriques et les résultats du travail de M. Morice nous confirment dans cette opinion, puisqu'ils ont permis d'aboutir à une détermination satisfaisante de l'élasticité de la demande du trafic postal et de son développement au cours du temps.

En ce qui concerne la période antérieure à la Guerre, le tarif ne fut modifié que deux fois; cette stabilité permet de considérer ces modifications de tarifs comme de véritables expérimentations, au sens de Cournot, puisque les périodes de tarif stable conduisent à une détermination satisfaisante de la loi de développement du trafic au cours du temps; il n'en est pas de même pour la période postérieure à la Guerre, et nous sommes alors contraints de recourir à un procédé purement statistique, dont la validité repose en grande partie sur les conclusions qui se dégagent de l'étude relative à l'avant-guerre.

Sans nous étendre davantage sur les résultats de cette étude, nous croyons cependant attirer l'attention sur les conclusions relatives à la

diminution du coefficient d'élasticité au cours du temps; ce phénomène peut s'expliquer à notre sens par l'interprétation que nous avons donné dans un travail antérieur[1] au sujet du coefficient d'élasticité: ce coefficient exprime en effet le rapport des consommations imparfaites à l'ensemble des consommations, les consommations imparfaites désignant celles pour lesquelles la satiété des besoins n'est pas réalisée. Il est bien certain en effet que le développement de la consommation d'un produit ou d'un service entraîne une réduction constante des consommations imparfaites rapportées à l'ensemble de la consommation, et par conséquent une diminution du coefficient d'élasticité. Si ce phénomène était vérifié par d'autres recherches, il serait peut-être possible, au cours d'un stade plus avancé, d'introduire dans l'équation de la demande, un coefficient d'élasticité variable en fonction du temps.

Université de Paris
Paris

[1] "La demande dans ses rapports avec la répartition des revenus," *Metron*, Vol. 8, 28 Fev. 1930, pp. 102–153.

LOI DE LA DEMANDE D'UN SERVICE MONOPOLISÉ

Par E. Morice

I. ÉTUDE GÉNÉRALE

La determination de la loi de la demande

$$\text{(I.1)} \qquad y = f(x, t), \qquad \begin{cases} y = \text{quantité demandée}, \\ x = \text{prix correspondant}, \\ t = \text{l'époque}, \end{cases}$$

à partir des observations statistiques (t_i, x_i, y_i) exige qu'on puisse considérer, à l'époque t_i, la consommation y_i comme étant en équilibre avec le prix moyen x_i de la période.

Ainsi que le fait remarquer M. Schultz, il y a lieu de "considérer la période élémentaire Δt, au cours de laquelle des consommateurs adaptent leurs achats aux prix nouveaux, comme une grandeur finie (année, mois, · · ·) dépendant de la nature du problème étudié.

"L'élasticité de la demande est fonction de l'intervalle de temps entre les observations successives à partir desquelles la courbe de la demande est construite."

Pour les denrées de consommation courante, dont les prix varient en général de façon continue et qui, d'autre part, sont soumises à des influences saisonnières importantes, agissant sur la consommation et sur le prix, le choix de l'année comme période élémentaire paraît s'imposer.

Mais si on étudie certains services monopolisés, les phénomènes ne se présentent plus de la même façon: les variations de tarifs sont peu fréquentes mais en général très importantes.

Ainsi qu'il résulte des études de M. Roy (consommation du gaz d'éclairage à Paris, variation des tarifs postaux) on constate alors que "l'action de la réduction de tarif se produit non seulement au moment même de la réduction de tarif, mais aussi pendant les années qui suivent," cette période d'influence étant, par exemple, de l'ordre de cinq ans pour la diminution du tarif du gaz d'éclairage, ramené de 0,30 fr. à 0,20 fr. en 1903.

Dans ces conditions, la loi (I.1) représentant l'ensemble des situations d'équilibre au cours du temps devra être ajustée à l'ensemble des observations (x_i, y_i, t_i), exception faite de celles qui se trouvent à l'intérieur de la période d'influence.

Le coefficient $\lambda = (d \log y/d \log x)$, calculé à partir de cette loi, sera le coefficient d'élasticité correspondant à la durée de la période d'influence.

Détermination approchée du coefficient d'élasticité

Dans l'hypothèse où les observations d'équilibre peuvent être adjustées par une loi de la forme

$$(\text{I.2}) \qquad y = Ae^{bt}p^{\lambda}, \qquad \begin{cases} y = \text{quantité}, \\ p = \text{prix}, \end{cases}$$

cherchons comment on peut déduire simplement le coefficient λ d'un groupe d'observations encadrant la période d'influence d'une modification de tarif.

Soient t_1 et t_2 deux points d'équilibre situés, l'un immédiatement avant le changement de tarif, l'autre à la fin de la période d'influence. Si p_1 et p_2 sont les prix correspondants, on a:

$$(\text{I.3}) \qquad \begin{aligned} \log_e y_1 &= \log_e A + bt_1 + \lambda \log_e p_1, \\ \log_e y_2 &= \log_e A + bt_2 + \lambda \log_e p_2, \end{aligned}$$

d'où

$$(\text{I.4}) \qquad \lambda = \frac{\log_e (y_2/y_1) - b(t_2 - t_1)}{\log_e (p_2/p_1)},$$

ou bien

$$(\text{I.5}) \qquad \lambda = \frac{\log_{10} y_2 - \log_{10} y_1 - b'(t_2 - t_1)}{\log_{10} p_2 - \log_{10} p_1},$$

dans laquelle b' représente l'accroissement moyen annuel de $\log_{10} y$ au cours des deux périodes qui encadrent la période d'influence.

Cette formule, subordonnée à l'existence d'une loi de la demande de la forme (I.2), n'a évidemment de sens que si l'accroissement annuel de $\log_{10} y$ reste sensiblement constant au cours des deux périodes d'équilibre.

Dans ce cas, si on a représenté les observations à l'aide d'un diagramme logarithmique et si les observations d'équilibre sont sensiblement alignées sur deux droites parallèles, la formule (I.5) peut s'écrire

$$(\text{I.6}) \qquad \lambda = \frac{\overline{N'N}}{\log_{10} p_2 - \log_{10} p_1},$$

$N'N$ étant la distance des deux droites, mesurée parallèlement à l'axe des log y (Figure 1).

En réalité, dans l'hypothèse où la loi de la demande est convenablement représentée par la formule (I.2), les observations seront simplement réparties au voisinage des deux droites M_1M et NN_1.

Il en résulte qu'utiliser la formule (I.5) c'est admettre qu'il faut utiliser l'ensemble des observations pour déterminer la direction de ces droites, mais qu'il suffit ensuite d'une observation isolée pour déterminer la position de chacune d'elles.

Or les erreurs commises sur y_1 et y_2—erreurs d'observation ou variations accidentelles des y—peuvent avoir une action importante sur les valeurs de λ.

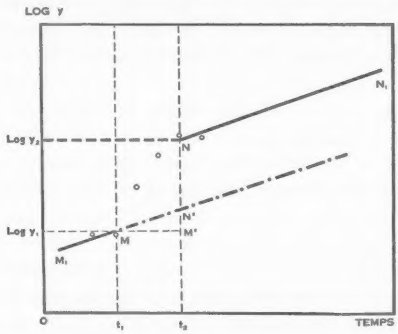


FIGURE 1.—0–t_1=période de prix invariable, p_1; t_1–t_2=période d'influence; t_2–=période de prix nouveau, p_2. Équation de M_1M: $\log y = \log A + b't + \lambda \log p_1$; équation de NN_1: $\log y = \log A + b't + \lambda \log p_2$. $M'N' = b'(t_2 - t_1)$; $N'N = \lambda \log (p_2/p_1)$.

Soient y_1' et y_2' les valeurs de y_1 et y_2 fournies par les droites ajustées. Posons

$$y_1 = (1+\alpha)y_1',$$
$$y_2 = (1+\beta)y_2',$$

on a

$$\log_e y_2 - \log_e y_1 = \log_e y_2' - \log_e y_1' + (\beta - \alpha) - \tfrac{1}{2}(\beta^2 - \alpha^2) + \cdots .$$

L'erreur commise peut donc être de l'ordre de

$$\frac{|\alpha| + |\beta|}{2{,}30[\log_{10} p_2 - \log_{10} p_1]} .$$

Ainsi pour $(p_2/p_1) = 3/5$ (modification des tarifs postaux de 1878), deux erreurs, de sens contraires, de 2 pour cent sur chacun des nombres y_1 et y_2 entraîneraient une erreur absolue de 0,08 sur λ dont la valeur est voisine de 0,60.

Si l'on veut seulement une valeur approchée de λ on pourra se contenter d'un ajustement graphique en diagramme logarithmique $(t, \log_{10} y)$ et utiliser (I.6).

Une autre question se pose: quelle durée faut-il donner à la période d'étude de $2n$ années encadrant la période d'influence? Si le taux d'accroissement annuel était rigoureusement constant, le résultat du calcul serait indépendant de cette durée. Mais il n'en sera qu'approximative-

ment ainsi: dans ces conditions, utiliser une période courte, c'est risquer d'attribuer une importance exagérée à certaines variations accidentelles des y; d'autre part prendre une période trop longue c'est risquer d'inclure dans la valeur calculée de λ des variations dûes à une modification systématique de la tendance séculaire, qui dissimule, en réalité, un grand nombre d'influences extérieures plus ou moins difficiles à caractériser.

En particulier, il y a lieu de songer à l'influence des variations cycliques sur la tendance apparente; la période d'étude devra être choisie de façon à rendre cette influence minimum (utilisation de cycles complets avant et après la période d'adaptation à la modification de tarif).

Dans chaque cas particulier, l'examen du tableau des différences premières de log $y(\Delta \log y_i = \log y_{i+1} - \log y_i)$ fournira d'utiles renseignements; à défaut de réponse formelle à la question ci-dessus, il y aura lieu d'examiner numériquement l'influence sur la valeur de λ d'une modification de la période d'étude: il suffira pour cela d'étudier l'influence des variations de b' = valeur moyenne de $\Delta \log y_i$ sur la formule (I.5); d'autre part, la durée même de la période d'influence n'est pas exactement connue, mais, si la tendance séculaire est nettement caractérisée, les résultats du calcul ne seront pas modifiés de façon sensible, si l'on commet une erreur par excès sur l'estimation de cette période.

II. ÉTUDE DU TRAFIC POSTAL, PÉRIODE 1873–1913

Les observations utilisées proviennent:

1°) Pour la période 1873–1884, de l'ouvrage de M. Belloc, *Les Postes françaises* (Firmin Didot, Éditeur, 1886).

2°) Pour la période 1876–1913, de la Statistique générale de Berne, recueil de documents publiés par l'Union postale internationale dont la France fait partie depuis 1876.

Ces deux statistiques sont très sensiblement concordantes pour la période commune 1876–1884 et donnent sous la rubrique "lettres ordinaires" les nombres annuels des lettres ordinaires et recommandées, paquets clos, lettres et paquets valeur déclarée.

L'origine des observations à été fixée au 1er Janvier 1873: 1° pour éliminer l'influence de la guerre de 1870–71; 2° pour tenir compte de la création des cartes postales ordinaires en Décembre 1872.

Au cours de la période 1873–1913 on enregistre deux modifications de tarif:

1°) Le 6 Avril 1878, le tarif des lettres ordinaires est ramené de 0,25 fr. à 0,15 fr.

2°) Le 6 Mars 1906, le tarif est de nouveau réduit de 0,15 fr. à 0,10 fr.

La documentation utilisée ne permet pas de tenir compte:

1°) du fait qu'avant 1875 le tarif distingue les lettres envoyées dans la circonscription d'un bureau de poste (tarif 0,15 fr.), des lettres envoyées de bureau à bureau (tarif 0,25 fr.)

2°) du fait qu'en 1910, un décret remplace les tarifs proportionnels prévus par les règlements postaux de 1875, 1878, et 1906 par un tarif dégressif, le poids maximum des lettres taxées à 0,10 fr. étant, de plus, porté de 15 à 20 grammes.

L'examen des relevés annuels détaillés publiés par le Ministère des P.T.T. pour la période 1922–1936 permet d'estimer que cette dernière modification n'a pas beaucoup d'influence sur la trafic total des lettres.

Les observations utilisées sont relevées dans le tableau suivant (tableau N° 1) qui donne aussi, pour chaque année, le logarithme du nombre de lettres.

L'examen du graphique N° I montre que:

1°) En 1878 comme en 1906, l'influence de la modification de tarif se fait sentire pendant une assez longue période: environ 4 ans en 1878, 2 à 3 ans en 1906. Le public met un certain temps à s'adapter aux variations brutales des tarifs.

2°) Lorsque l'équilibre est établi, le tarif restant invariable, les points correspondant aux observations sont sensiblement alignés sur trois droites, $A'A$, BB', CC', de même pente ce qui conduit à envisager une loi de la forme

$$\log N = a + bt + f(p),$$

$f(p)$ étant une fonction du tarif p appliqué au cours de la période.

3°) De plus, si on compare les accroissements de trafic dûs aux variations de tarif, la tendance séculaire étant éliminée, on constate que les deux rapports

$$\frac{AB}{\log p_2 - \log p_1} \quad \text{et} \quad \frac{B'C'}{\log p_3 - \log p_2}$$

sont assez voisins, c'est-à-dire que le rapport

$$\frac{\log N_2 - \log N_1 - b(t_2 - t_1)}{\log p_2 - \log p_1}$$

reste sensiblement constant pendant toute la période étudiée

En première analyse on est donc conduit à une loi de la demande, à élasticité constante, de la forme classique:

$$N = A \times 10^{b't} \times p^{\lambda}, \tag{II.1}$$

λ étant le coefficient d'élasticité.

TABLEAU N° I

Années		Nombre de lettres N (millions)	Log N	Différences (log N) $\times 10^4$	Tarif en C^{mes}	Nombre de cartes postales ordinaires (millions)	Tarif en C^{mes}
0	1873	285	2,4548	106	25	16	15
1	4	292	4654	189	25	16	15
2	5	305	4843	85	25	21	15
3	6	311	4928	123	25	26	15
4	7	320	5051	619	25	31	15
5	8	369	5670	351	15	30	10
6	9	400	6021	354	15	26	10
7	1880	434	6375	446	15	28	10
8	1	481	6821	81	15	30	10
9	2	490	6902	165	15	31	10
10	3	509	7067	60	15	31	10
11	4	516	7127	58	15	32	10
12	5	523	7185	74	15	38	10
13	6	532	7259	97	15	34	10
14	7	544	7356	149	15	36	10
15	8	563	7505	84	15	38	10
16	9	574	7589	207	15	42	10
17	1890	602	7796	93	15	43	10
18	1	615	7889	146	15	44	10
19	2	636	8035	34	15	45	10
20	3	641	8069	146	15	46	10
21	4	663	8215	148	15	46	10
22	5	686	8363	07	15	49	10
23	6	687	8370	203	15	49	10
24	7	720	8573	149	15	5	10
25	8	745	8722	149	15	53	10
26	9	771	8871	182	15	55	10
27	1900	804	9053	−44	15	57	10
28	1	796	9009	150	15	61	10
29	2	824	9159	104	15	64	10
30	3	844	9263	87	15	70	10
31	4	861	9350	85	15	74	10
32	5	878	2,9435	630	15	77	10
33	6	1015	3,0065	431	10	40	10
34	7	1121	0496	156	10	21	10
35	8	1162	0652	70	10	18	10
36	9	1181	0722	106	10	16	10
37	1910	1210	0828	50	10	15	10
38	1	1224	0878	91	10	15	10
39	2	1250	0969	237	10	14	10
40	3	1320	3,1206		10	14	10

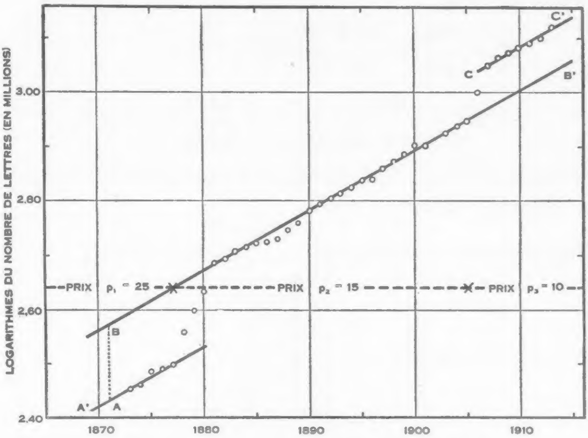
Si on pose

$$a = \log A,$$

l'ajustement de la formule

$$\log N = a + b't + \lambda \log p \tag{II.2}$$

aux données, fournirait la valeur des coefficients a, b, λ à condition d'ajuster la formule (II.2) à l'ensemble des observations d'équilibre, c'est-à-dire, à condition de négliger, pour l'instant, les observations situées à l'intérieur des périodes d'influence.



GRAPHIQUE No. I.—Trafic postal, 1873–1913.

Cependant une solution aussi simple ne paraît pas satisfaisante: en effet l'écart entre les deux valeurs

$$\frac{AB}{\log p_2 - \log p_1} = -\frac{0{,}135}{0{,}222} = -0{,}61,$$

et

$$\frac{B'C'}{\log p_3 - \log p_2} = -\frac{0{,}085}{0{,}176} = -0{,}48,$$

est certainement supérieur à l'erreur que l'on peut craindre en employant la méthode graphique.

Il semble plus plausible d'admettre que l'élasticité à diminué de façon appréciable entre 1878 et 1906.

On est ainsi conduit à examiner séparément les deux modifications de tarif.

A. Modification de tarif de 1878

Le graphique N° II, relatif à la période 1873–1895, montre que

1°) une formule de la forme (II.2) s'adapte beaucoup mieux aux observations qu'une formule linéaire,

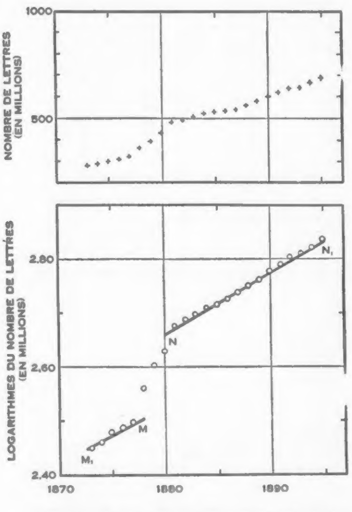
$$N = \alpha + \beta t + \lambda' p.$$

2°) Si on trace, au mieux, deux droites parallèles correspondant respectivement aux deux ensembles d'observations d'équilibre 1873–1877 et 1881–1895, le graphique permet de déterminer rapidement les valeurs approchées

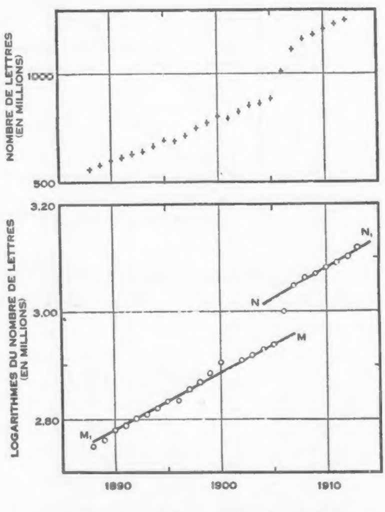
$$b' = \frac{PM}{AP} = \frac{0{,}11}{10} = 0{,}011,$$

$$\lambda = \frac{MM'}{\log 15 - \log 25} = -\frac{0{,}135}{0{,}222} = -0{,}61,$$

caractérisant un taux d'accroissement annuel d'environ 2,5 pour cent et une élasticité voisine de $-0{,}60$.



Graphique No. II.—Modification de tarif de 1878. Équation de M_1M: log $N = a + 0{,}011t - 0{,}59$ log 25; équation de NN_1: log $N = a + 0{,}011t - 0{,}59$ log 15.



Graphique No. III.—Modification de tarif de 1906. Équation de M_1M: log $N = a + 0{,}011t - 0{,}44$ log 15; équation de NN_1: log $N = a + 0{,}011t - 0.44$ log 10.

L'adjustement de la formule (2) par la méthode des moindres carrés ne semble pas devoir améliorer beaucoup la précision des résultats. Il permet cependant de chiffrer l'accord entre la droite ajustée et les données numériques à l'aide de l'écart quadratique moyen difficile à évaluer sur le graphique.

Cet ajustement donne, pour les années 1873–1877 et 1881–1895, la formule

$$\operatorname{Log} N = 3{,}28071 + 0{,}01104t - 0{,}58822 \log p,$$

$$(t = 0 \text{ en } 1873,\ p = \text{prix en centimes}),$$

$$\text{soit} \begin{cases} b' = 0{,}011, \\ \lambda = 0{,}59. \end{cases}$$

Le tableau ci-joint (tableau N° II) montre la précision de l'ajustement, l'écart le plus grand est de 1,83 pour cent, l'écart quadratique moyen $\sigma = 5{,}27$, rapporté à la valeur moyenne des observations, vaut 1,1 pour cent.

Tableau N° II

Années	Observations N	Calcul N'	Ecarts $N'-N$		
			−	+	%
1873	285	287,3		2,3	0,81
4	292	294,7		2,7	0,92
5	305	302,3	2,7		0,89
6	311	310,1	0,9		0,29
7	320	318,1	1,9		0,60
1878–1880					1,1
1881	481	475,6	5,4		1,14
2	490	487,8	2,2		0,45
3	509	500,4	8,6		1,68
4	516	513,3	2,7		0,57
5	523	526,5		3,5	0,67
6	532	540,1		8,1	1,50
7	544	554,0		10,0	1,83
8	563	568,2		5,2	0,92
9	574	582,8		8,8	1,53
1890	602	597,8	4,2		0,67
1	615	613,2	1,8		0,29
2	636	629,0	7,0		1,10
3	641	645,2		4,2	0,65
4	663	661,9	1,1		0,17
5	686	678,9	7,1		1,03

Substitution des cartes postales

Avant 1878, le public dispose comme moyens de correspondance de la lettre à 25 centimes et de la carte postale ordinaire à 15 centimes. En 1878 ces deux tarifs sont ramenés respectivement à 15 et 10 centimes.

Il se produit alors un phénomène de substitution, mis en évidence par le tableau suivant:

Années	1873	1874	1875	1876	1877	1878	1879	1880	1881	1882	1883	1884	1885
Cartes postales (millions)	16	16	21	26	31	30	27	28	30	31	32	33	38

Le nombre des cartes postales transformées en lettres au cours des années 1878, 1879, · · · est de l'ordre de 5 millions la 1ère année, 10 à 15 millions l'année suivante · · · Mais il n'est guère possible de tenir compte dans le calcul d'une telle extrapolation, d'autant plus que la tendance séculaire du trafic des cartes postales est déterminé à l'aide d'un trop petit nombre d'années (la carte postale ordinaire n'est en service que depuis 1873).

A titre d'indication, signalons seulement qu'en tenant compte des résultats de cette extrapolation jusqu'en 1881, le coefficient d'élasticité des lettres ordinaires se trouverait reduit de $-0,60$ à $-0,55$, ce résultat permettant tout au plus de conclure que le phénomène de substitution est relativement peu important en 1878.

B. Modification de tarif de 1906

Le graphique N° III représente les variations du logarithme du nombre N de lettres ordinaires pour la période 1892–1913.

La méthode des moindres carrés appliquée à une formule de la forme (II.2) pour l'ensemble des observations d'équilibre 1892–1905, 1908–1913, donne

$$\log N = 3{,}31635 + 0{,}01132t - 0{,}4370 \log_{10} p,$$

dans laquelle p est le prix en centimes et $t=0$ en 1892.

Cette formule caractérise un taux d'accroissement annuel d'environ 2,6 pour cent et une élasticité voisine de $-0{,}44$.

Le tableau ci-joint (Tableau N° III) donne les résultats de cet ajustement: l'écart le plus grand est de 2,90 pour cent, l'écart quadratique moyen $\sigma=11{,}3$, rapporté à la valeur moyenne des observations vaut 0,58 pour cent.

Cependant un examen du graphique montre que la tendance séculaire a été moins stable qu'au cours de la période précédemment étudiée: entre 1896 et 1900 elle est manifestement plus importante, environ 4 pour cent.

Cette modification semble liée à la période d'essor constatée à cette époque dans l'ensemble de l'économie française.

La période étudiée se termimant nécessairement en 1913, en raison de la guerre, il n'est pas possible de choisir cette période d'étude de façon qu'elle comprenne un même nombre de cycles de part et d'autre de la période d'influence.

Dans ces conditions, il se peut que les résultats précédents soient largement influencés par les observations de la période 1896–1900.

TABLEAU N° III

Années	Observations N	Calcul N'	Ecarts $N'-N$		
			−	+	%
1892	636	634,5	1,5		0,25
3	641	651,2		10,2	1,57
4	663	668,5		5,5	0,83
5	686	686,1		0,1	0,01
6	687	704,2		17,2	2,45
7	720	722,8		2,8	0,39
8	745	741,9	3,1		0,42
9	771	761,5	9,5		1,25
1900	804	781,6	22,4		2,87
1	796	802,2		6,2	0,78
2	824	823,4	0,6		0,07
3	844	845,1		1,1	0,13
4	861	867,5		6,5	0,76
5	878	890,3		12,3	1,39
1906–1907					
8	1162	1149	13		1,13
9	1180	1180	—	—	—
1910	1210	1211		1	0,08
1	1224	1243		19	1,53
2	1250	1276		26	2,04
3	1320	1309	11		0,84

Pour se rendre compte de l'influence de cette modification de la tendance sur le coefficient d'élasticité, on peut envisager l'étude d'une période d'équilibre plus courte, par exemple, la période 1901–1905, 1908–1912.

On trouve ainsi les nouvelles valeurs $b'=0{,}00935$, $\lambda=0{,}5186$ caractérisant un taux d'accroissement annuel d'environ 2,15 pour cent et une élasticité assez différente de celle calculée ci-dessus.

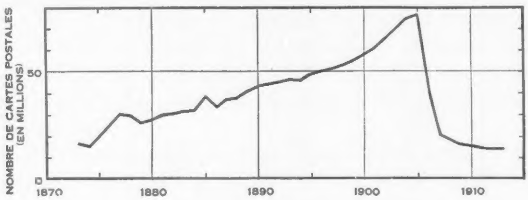
On peut estimer que le coefficient cherché se trouve dans l'intervalle ainsi déterminé, c'est-à-dire de l'ordre de $-0{,}50$.

Substitution des cartes postales

Ce phénomène a été beaucoup plus important qu'en 1878.

En effet, en 1906, le tarif des lettres passe de 0,15 fr. à 0,10 fr., mais le tarif des cartes postales ordinaires reste fixé à 0,10 fr., d'où une très importante diminution du trafic des cartes postales ordinaires au cours des années 1906 et 1907. Ce phénomène est illustré par le graphique N° IV.

L'extrapolation, toujours délicate, est ici particulièrement difficile: en effet le trafic des cartes postales ordinaires, toujours croissant de 1880, à 1905 reste constamment décroissant de 1905 à 1913. Il semble que la modification de tarif, dont l'influence sur le trafic des cartes postales a été particulièrement importante en 1906 et 1907, continue à agir au cours des années suivantes, ne permettant pas de déterminer une situation d'équilibre correspondant aux nouveaux tarifs.



GRAPHIQUE No. IV.—Cartes postales ordinaires, 1873–1913.

Pour ne tenir compte que des variations les plus importantes on peut estimer approximativement à 60 millions, le nombre total de cartes postales ordinaires transformées en lettres en 1908, ce qui donne pour cette année, le nombre corrigé

$$y_2' = y_2 - 60 = 1162 - 60 = 1102.$$

Cette correction, appliquée intégralement à la formule (I.5), aurait pour effet, en négligeant son influence, d'ailleurs petite, sur la tendance moyenne b', de réduire d'environ 25 pour cent la valeur de λ fournie par cette formule, la ramenant de $-0{,}53$ à $-0{,}40$.

En raison de l'imprécision des éléments de ce calcul, il est difficile de déterminer la valeur du coefficient d'élasticité des lettres seules pour la période étudiée.

Il est cependant intéressant de constater que le coefficient probablement compris entre $-0{,}35$ et $-0{,}45$ est certainement inférieur, en valeur absolue, au coefficient d'élasticité de 1878, voisin de $-0{,}60$.

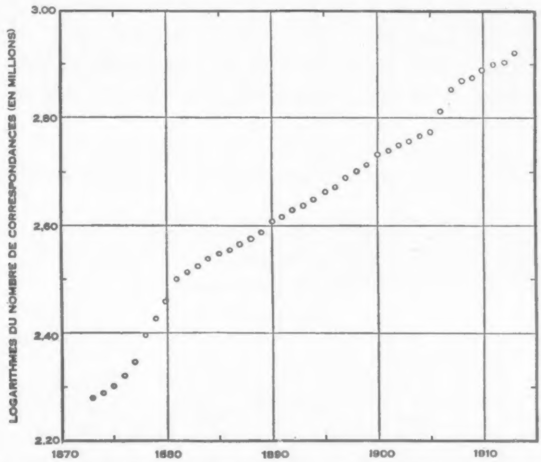
Étude d'ensemble des lettres et cartes postales ordinaires

Les statistiques utilisées contiennent sous la rubrique "lettres ordinaires" non seulement les lettres dont le poids est inférieur à 20 grammes et auxquelles s'applique le tarif envisagé, mais encore les lettres et paquets clos ordinaires pesant de 20 grammes à 1500 grammes auxquels s'applique un tarif progressif.

L'absence de documents statistiques ne permet pas de tenir compte de la répartition par catégories. Constatons simplement qu'après

guerre, le nombre de ces lettres et paquets représente environ 10 pour cent du nombre total des lettres. Au cours de la période 1873–1913, le nombre des cartes postales ordinaires représente 5 à 8 pour cent du nombre des lettres.

Dans ces conditions, il semble plausible, dans les mêmes limites d'approximation, d'étudier simultanément le trafic des lettres et des cartes postales ordinaires, l'ensemble restant caractérisé au point de vue prix, par le tarif des lettres ordinaires pesant moins de 20 grammes (Graphique N° V).



GRAPHIQUE No. V.—Lettres et cartes postales ordinaires, 1873–1913.

Dans cette hypothèse, la formule (I.5) donne les résultats suivants:

1°) Modification de 1878: $\lambda = -0{,}50$, les observations ayant servi à déterminer la tendance b', étant celles des périodes 1873–1877 et 1881–1885.

2°) Modification de 1906: $\lambda = -0{,}35$, les observations utilisées étant celles des périodes 1901–1905 et 1908–1912.

Ces calculs, malgré leur imprécision, montrent une fois de plus, que le coefficient d'élasticité caractérisant le besoin étudié a diminué en valeur absolue entre 1870 et 1910.

III. PÉRIODE 1922–1936

Après la guerre 1914–1918, la dépréciation de la monnaie rend nécessaires plusieurs majorations de tarif, se succédant assez rapidement de 1920 à 1926.

L'affranchissement des lettres ordinaires fixé à 0,15 fr. depuis le 30 Décembre 1916 est porté successivement à:

0,25 fr. le 29 Mars 1920,

0,30 fr. le 18 Juillet 1925,

0,40 fr. le 29 Avril 1926,

0,50 fr. le 5 Août 1926; il reste fixé à ce tarif jusqu'en 1937.[1]

Le tarif des cartes postales ordinaires subit des modifications analogues:

0,15 fr. de 1916 à 1920,

0,20 fr. le 29 Mars 1920,

0,30 fr. le 29 Avril 1924,

0,40 fr. le 5 Août 1926.

Les données statistiques relatives à la période 1922–1936 sont résumées dans le tableau suivant (Tableau N° IV) qui donne pour chaque année:

1°) le nombre de lettres ordinaires, en millions, N;

2°) le tarif moyen annuel, compte tenu de la date du changement de tarif.

3°) le tarif réel, p, calculé en utilisant l'indice des prix de détail de la Statistique générale de la France.

4° et 5°), les valeurs de $\log_{10} N$ et de $\log_{10} p$.

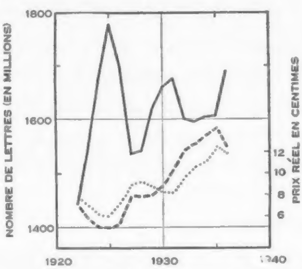
TABLEAU N° IV

Année	t	Nombre de lettres (millions) N*	Tarif moyen Cmes	Indice des prix de détail	Prix réel p	$\text{Log}_{10} N$	$\text{Log}_{10} p$
1922	0	1436	25	318	7,86	3,1571	0,8954
1923	1	1540	25	349	7,16	3,1875	0,8549
1924	2	1670	25	407	6,14	3,2227	0,7882
1925	3	1776	27,5	455	6,04	3,2494	0,7810
1926	4	1702	40,6	571	7,11	3,2310	0,8519
1927	5	1536	50	559	8,94	3,1864	0,9513
1928	6	1543	50	537	9,31	3,1884	0,9689
1929	7	1618	50	583	8,58	3,2090	0,9335
1930	8	1663	50	607	8,24	3,2209	0,9159
1931	9	1674	50	613	8,16	3,2238	0,9117
1932	10	1601	50	532	9,40	3,2044	0,9731
1933	11	1593	50	475	10,53	3,2022	1,0224
1934	12	1605	50	455	10,99	3,2055	1,0410
1935	13	1609	50	395	12,66	3,2066	1,1024
1936	14	1687	50	434	11,52	3,2271	1,0614

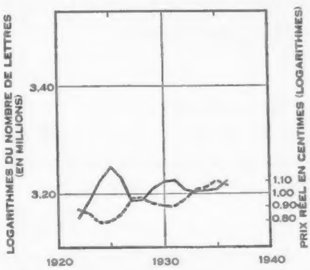
* Ces nombres—extraits des relevés annuels détaillés publiés par le Ministère des P.T.T.—comprennent, non seulement le service intérieur, mais aussi le service international.

[1] Le décret du 12 Juillet 1937 porte respectivement à 0,65 fr. et 0,55 fr. les tarifs des lettres ordinaires et des cartes postales ordinaires.

Les graphiques N° VI et VII qui illustrent ce tableau montrent qu'à l'augmentation de tarif de 1926 répond bien une diminution du nombre des letters, diminution qui s'est d'ailleurs accentuée en 1927. Par contre, malgré la stabilité du tarif nominal entre 1927 et 1933, on constate d'importantes variations du nombre de lettres qui croît de 1536 millions en 1927 à 1674 millions en 1931 pour décroître à 1593 millions en 1933.



Graphique No. VI.————: Nombre de lettres (en millions); - - - - : Prix réel (Indice des prix de gros); · · · · : Prix réel (Indice des prix de detail).



Graphique No. VII.————: Logarithmes du nombre de lettres (en millions); - - - - : Prix réel en centimes (logarithmes).

Dans ces conditions, il ne saurait être question de chercher une relation entre le nombre de lettres, le tarif nominal, et le temps: ce serait dissimuler sous l'étiquette commode "fonction du temps" une des causes essentielles de la variation du nombre des lettres.

Pour la même raison, il est illusoire de chercher une valeur approchée de l'élasticité en comparant deux valeurs du nombre des lettres avant et après une modification de tarif nominal; on ignore, en effet, quelle est la part de la variation globale du nombre des lettres qui est imputable à la variation de tarif.

Parmi les causes, évidemment nombreuses et complexes, qui peuvent agir sur le trafic postal, deux attirent tout spécialement l'attention:

1°) les variations du "tarif réel" dont le grand public se rend compte par la comparaison des variations de prix des diverses denrées de consommation, même s'il ignore les index de prix et leur signification précise.

2°) les variations de l'activité économique qui agissent évidemment sur le nombre des correspondances commerciales.

S'il est facile de tenir compte de la première de ces causes, l'action de la seconde est beaucoup plus difficile à caractériser, faute de connaître la part du trafic postal correspondant aux lettres d'affaires.

Étude de l'influence des variations du tarif réel sur le trafic postal

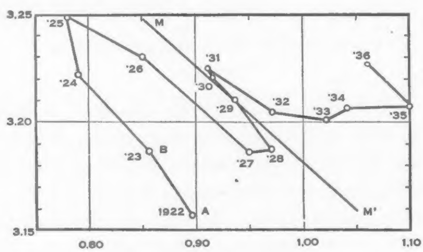
Les graphiques N° VI et VII montrent qu'en général à une augmentation (ou diminution) du tarif réel correspond une diminution (ou augmentation) du trafic postal. Les exceptions constatées—correspondant d'ailleurs à des variations de faible amplitude—se présentent en période d'augmentation du tarif réel et peuvent ainsi s'expliquer par l'action restée croissante de la tendance séculaire constatée avant guerre.

Ajustement par une courbe d'élasticité constante

La méthode employée est la méthode de *régression temporelle* utilisée par M. Henry Schultz, pour l'étude de l'élasticité de la demande d'un assez grand nombre de produits agricoles (blé, maïs, pomme de terre, orge, sucre).

1°) Étude graphique

Le graphique N° VIII montre le diagramme de dispersion des observations (log p, log N).



GRAPHIQUE No. VIII.

On constate que les segments tels que AB joignant les observations de deux années consécutives ont, en général, une pente négative caractérisant le fait que, pour une courte période les variations de prix réel et de quantité sont de sens inverses.

De plus, pour un même prix origine (ou des prix voisins), les divers segments ont tendance à s'élever depuis 1922 jusqu'à 1936, manifestant ainsi une tendance séculaire croissante.

Si on trace une droite MM', passant par le point moyen des deux séries et ayant pour direction la direction moyenne des segments déterminée approximativement, la pente de cette droite fournit une valeur approchée du rapport des accroissements correspondants de log N et de log p, c'est-à-dire de l'élasticité.

L'étude de la répartition dans le temps, des écarts d'ordonnées entre les points correspondant aux observations et la droite MM', permet d'étudier le déplacement de la courbe de demande au cours du temps, c'est-à-dire de résumer l'influence permanente de causes diverses (évolution des habitudes, chiffre de population, · · ·) agissant indépendamment du prix.

Le graphique N° IX représente la variation des écarts. Les points représentatifs sont distribués sensiblement au voisinage d'une droite PP' dont la pente caractérise la tendance séculaire de log N, c'est-à-dire l'accroissement annuel moyen de log N, lorsque le prix réel est supposé constant.

On est ainsi conduit à ajuster aux données une équation de la forme (II.2).

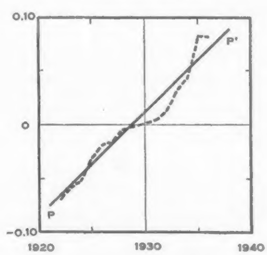
L'ajustement, par la méthode des moindres carrés, donne

$$\log N = 3{,}52869 + 0{,}008704t - 0{,}40719 \log p,$$

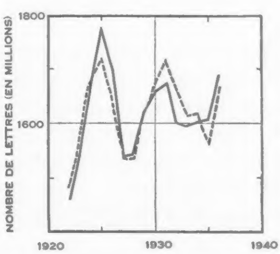
$$(t = 0 \text{ en } 1922,\ p = \text{prix réel en centimes}).$$

Cette formule met en évidence un taux d'accroissement annuel d'environ 2 pour cent et une élasticité constante voisine de $-0{,}41$.

Le tableau ci-joint (Tableau N° V) donne les résultats de cet ajustement: l'écart le plus grand est de 3,6 pour cent, l'écart quadratique moyen $\sigma = 34$, rapporté à la valeur moyenne des observations vaut 2,10 pour cent (voir graphique N° X).



Graphique No. IX.



Graphique No. X.—Nombres de lettres (en millions):————: Observations; - - - - : Valeurs calculées.

Le graphique N° IX montre qu'on obtiendrait un meilleur ajustement de la tendance séculaire à l'aide d'une fonction du troisième degré, mais les écarts indiqués cidessus sont assez petits pour que la détermination d'une telle fonction semble illusoire.

D'autre part, la distribution quelque peu systématique des écarts, par excès de 1925 à 1928, par défaut de 1929 à 1934, ne semble pas

dûe uniquement au hasard, peut-être ces écarts sont-ils, dans une certaine mesure, liés aux variations de la situation économique générale. L'examen de divers indices publiés par la Statistique générale de la France ne permet pas de conclure d'une façon formelle.

Tableau N° V

Année	Observations N	Calcul N'	Ecarts $N'-N$		
			–	+	%
1922	1436	1459		23	1,6
3	1540	1546		6	0,4
4	1670	1679		9	0,5
5	1776	1725	51		2,9
6	1702	1647	55		3,2
7	1536	1531	5		0,4
8	1543	1536	7		0,5
9	1618	1620		2	0,1
1930	1663	1680		17	1,
1	1674	1721		47	2,8
2	1601	1658		57	3,6
3	1593	1615		22	1,4
4	1605	1619		14	0,9
5	1609	1559	50		3,1
6	1687	1653	34		2,

Influence des prix antérieurs

L'étude de la période d'avant-guerre a montré que l'influence d'une modification importante se faisait sentir pendant plusieurs années, on peut exprimer ce fait en admettant que le trafic au cours d'une année est fonction, non seulement du tarif actuel, mais encore du tarif des deux ou trois années précédentes.

Ceci conduit, dans l'hypothèse d'une tendance séculaire caractérisée par un taux d'accroissement constant, à envisager une loi de la forme

$$y_n = A \cdot 10^{b't} \overset{\alpha}{p_n} \overset{\beta}{p_{n-1}} \cdots, \tag{I.2}$$

y^n représentant le trafic de la n^{ie} année, $p_n, p_{n-1}, \cdots$ les prix correspondant respectivement à la n^{ieme} année et aux années précédentes.

Cependant, malgré la présence d'un paramètre supplémentaire, la formule

$$\log y_n = a + b't + \alpha \log p_n + \beta \log p_{n-1}$$

ne donne pas un meilleur ajustement que la formule (II.2).

CONCLUSIONS

1) Le service du transport des correspondances privées existe sous des formes variées depuis très longtemps, mais il n'est vraiment devenu un service mis à la disposition du grand public que depuis le milieu du siècle dernier.

Les calculs précédents montrent que le coefficient d'élasticité a diminué, en valeur absolue, au cours de la période 1870–1910, d'environ 0,60 à 0,40: le service considéré satisfaisant un besoin de plus en plus urgent.

L'élasticité correspondant à la période d'après guerre est sensiblement la même que celle relative à la période 1900–1913: le besoin de correspondre semble s'être stabilisé dans la hiérarchie des besoins.

2) D'autre part la tendance séculaire est mesurée par un taux d'accroissement sensiblement constant de 2,5 pour cent au cours de la période 1870–1910 caractérisée par le développement progressif du niveau général d'instruction, la facilité croissante des voyages, et l'extension des courants commerciaux.

Ce taux d'accroissement semble diminuer légèrement après guerre = 2 pour cent au lieu de 2,5 pour cent pour un tarif réel supposé invariable. Constatons cependant que le taux d'accroissement de la période 1922–1936 est notablement supérieur au taux moyen 1,05 pour cent de la période 1913–1923 (influence de la guerre).

3) Ainsi que M. Schultz l'a noté dans son étude de la demande du sucre aux États-Unis, on constate que les diverses méthodes employées donnent sensiblement la même valeur du coefficient d'élasticité moyen relatif à la période 1922–1936 et ceci, en particulier, malgré les variations annuelles fort différentes des coefficients fournis par une formule linéaire appliquée soit à la méthode de régression temporelle, soit à la méthode des chaînes de rapports.

En raison de la variabilité de ces coefficients annuels qui, dans un domaine d'observations nécessairement discontinués sont trop dépendants des valeurs locales qui servent à les calculer, il semble préférable d'utiliser pour l'ensemble d'une période, une formule à élasticité constante donnant directement pour cette période la valeur moyenne que l'on pourrait déduire de ces valeurs annuelles.

4) En ce qui concerne l'élaboration des résultats, il semble que c'est la méthode des chaînes de rapports (formule logarithmique) qui se prête le mieux à une détermination graphique approchée du coefficient d'élasticité, c'est-à-dire du coefficient angulaire de la droite autour de laquelle se répartissent les points $(\log X_i, \log Y_i)$: $Y_i = N_i/N_{i-1}$; $X_i = p_i/p_{i-1}$.

5) L'étude de la période 1922–1936 montre, de la part du grand

public, une sensibilité très nette aux prix réels. Il n'a pas été possible de mettre en évidence les modifications de trafic provoquées par l'annonce d'une variation du tarif nominal considérée comme agissant indépendamment des variations du prix réel.

6) En ce qui concerne le rendement on constate que le produit global —évalué en fonction du prix réel envisagé cidessus—varie dans le même sens que ce tarif réel, quelles que soient les variations correspondantes de trafic.

De plus, pour un trafic restant au cours d'une période de 15 ans, compris entre 1440 et 1780 millions de lettres (écart inférieur à 25 pour cent), on constate que le produit global—évalué comme ci-dessus —a varié du simple au double: de 103 à 204 millions. Il semble, dans ces conditions qu'une adaptation mieux étudiée des tarifs postaux aux conditions économiques du moment et en particulier à la valeur de la monnaie, aurait sérieusement augmenté le rendement réel du monopole au cours des années 1922 à 1931, pendant lesquelles le tarif réel moyen est resté inférieur à 8 centimes (6 centimes en 1925), alors qu'il était de 10 centimes en 1913.

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MONEY AND THE THEORY OF ASSETS

By J. MARSCHAK

1. THE SCOPE

THERE has been until recently little connection between what is usually taught as Monetary Theory and the General Theory of Prices (or Value). Furthermore, there is little connection between the two compartments of Monetary Theory: the Theory of the "Equation of Exchange" with its underlying concepts, Price Level, Velocity of Circulation, Real Income, etc., on the one hand and, on the other, the Theory of Credit and Banking. The Babylonian lack of common language between the two compartments of Monetary Theory is well illustrated by the fact that, while "Exchange Value of Money" (Wicksell, Robertson) is defined as the reciprocal of the Price Level, the term Price of Money is often used to designate interest rates on short loans—although usually economists agree to use Exchange Value of a thing and Price, of a thing as equivalent terms. In neither of the two compartments of Monetary Theory is much use made of the fundamentals of economic theory. The Price Level is treated as if it had nothing to do with prices. The velocity of circulation has been, thanks to the Cambridge School, associated with cash holdings instead of being left in the air; but the next step, to treat cash holdings on the same lines on which holdings of any other Stocks are treated in the General Theory of Prices, has been made by few economists only, of whom Dr. Hicks is the most outstanding.[1] Similarly, the Interest Rate of the treatises on Banking and Trade Cycle seems to be ashamed of any connection with its less adventurous and "dynamic" but more scholarly relation, the Interest Rate of the marginal productivity theory: the parentage is casually mentioned, if at all, in a few hurried phrases only.

F. Divisia has suggested[2] to add the Equation of Exchange to the Paretian system of equations. This makes it possible to determine the absolute and not only the relative price of goods if the velocity of circulation is given. But we may refuse to regard the velocity of circulation as given because we want to find the factors determining it, or determining Walras' *encaisse desirée*, just as we look for the factors determining the desired quantity of houses, stocks of materials in the factory or in a housewife's larder, securities, bills, or any other balance-sheet items. We do not want the Velocity of Circulation to save the situation like a *deus ex machina*. Yet there is the difficulty that money

[1] *Economica*, 1935. See also the interesting article by P. Rosenstein-Rodan, *Economica*, 1936.

[2] *Économie Rationelle.*

is neither a production good nor a consumers' good, and the technique used to explain the causation of quantities and prices of such goods may not apply to money unless we find some way of generalizing the concepts used.

Recently, the rigid (i.e., given) velocity of circulation has been discarded, and the cash holdings expressed as a function of the interest rate, a kind of demand function. Mr. Keynes' "liquidity preference" is a great step forward, provided it is not taken to be simply another word for the old and mysterious "love of money"; provided that it yields itself to a further rational analysis.

The desire to treat monetary problems and indeed, more generally, problems of investment with the tools of a properly generalized Economic Theory is not merely one of aesthetics or disciplinarianism—although the example of the gradual unification of the various forms of Energy in Nineteenth Century Physics might in itself encourage the search for unity. The unsatisfactory state of Monetary Theory as compared with general Economics is due to the fact that the principle of determinateness so well established by Walras and Pareto for the world of perishable consumption goods and labour services has never been applied with much consistency to durable goods and, still less, to claims (securities, loans, cash). To do this requires, first, an extension of the concept of human *tastes:* by taking into account not only men's aversion for waiting but also their desire for safety, and other traits of behaviour not present in the world of perfect certainty as postulated in the classical static economics. Second, the *production conditions,* assumed hereto to be objectively given, become, more realistically, mere subjective expectations of the investors—and *all* individuals are investors (in any but a timeless economy) just as all market transactions are investments. The problem is: to explain the objective quantities of goods and claims held at any point of time, and the objective market prices at which they are exchanged, given the subjective tastes and expectations of the individuals at this point of time.

2. PROCEDURE[3]

Our procedure is as follows: we recapitulate the theory of economic determinateness[4] properly generalised so as to include the case of joint demand and supply (which will be found to be important later). We

[3] Cp. Makower and Marschak, *Economica,* 1938.

[4] We prefer this term to "economic equilibrium" as the stability of a given system of quantities *in time* is not implied. For brevity only, we shall occasionally use expressions like "the *equilibrium amount a* of an asset" to mean "the amount which satisfies the set of determining equations"; taking the point of view of the individual concerned we may also use, instead, the expression "the *best accessible amount* of an asset."

treat perfect competition as a special, imperfect competition as a general case. We then gradually extend the meaning of the symbols so as to take account of uncertainty and ignorance. At no stage do we assume a "timeless" economy—an assumption which has always made the economics of production extremely vague.

For each of the r individuals in the market there are two sets of unknowns: (1) his best accessible balance sheet, or best accessible collection of m present *assets* $a, b, \cdots$; and (2) his best accessible consumption plan, or best combination of n future consumption items $x, y, \cdots$ which we shall call for brevity "*yields*," because unfortunately "consumptions" would be ungrammatical. A further set of unknowns are (3) the m market *prices* of assets: p (of a), q (of b), etc.

The yields, or amounts consumed, $x, y, \cdots$, differ either in quality or with respect to the time interval in which the consumption takes place. Thus, x may be first year's milk consumption, y second year's milk consumption, z first year's meat consumption, etc. Yields cannot be stocked or exchanged. Assets are stocks of either goods or claims. Assets are thus a wider concept than "producers' goods" or indeed than any goods in stock. Yields, on the other hand, are not stocks of consumers' goods, but flows of amounts consumed. Assets are items (positive or negative) in a balance sheet; yields are items in a family budget. By buying and selling assets in the market the individual arranges his present balance sheet in the time point 1 so as to be able to enjoy the best accessible consumption plan, i.e., the most satisfactory series of family budgets for the time intervals 1–2, 2–3, etc. The yields $x, y, \cdots$ refer thus to the future. The assets $a, b, \cdots$ as well as their prices $p, q, \cdots$ refer to the present. It is not necessary to introduce here the concept of "future assets." Exchanges in the market of "futures" can be regarded as exchanges of present claims.

It is seen that the assets are "jointly demanded"; the significance of an asset to the investor depends on what other items he has on his balance sheet. And a given balance sheet enables him to receive not one yield but a whole set of mutually dependent yields; yields are "jointly supplied."

3. THE CRUSOE CASE

Certain rudimentary features of the system can be studied already in the case of an isolated producer-consumer. There are here n unknowns only: the yields $x, y, \cdots$. The assets $a, b, \cdots$ are fixed since there is no exchange, but the individual can choose between various consumption plans. His *tastes* are given by the utility function

$$U(x, y, \cdots)$$

(or any monotonically increasing function of U), which has to be maximized, subject to the restricting *transformation condition*

$$T(x, y, \cdots; a, b, \cdots) = 0, \tag{3.1}$$

which describes what consumption plans the individual expects to be accessible with the given asset collection. Equating to zero the complete differential of U, and comparing it to the complete differential of T, we obtain $n-1$ equations of the type

$$U_x/U_y = T_x/T_y, \text{ (yield-distribution equation),} \tag{3.2}$$

which together with (3.1) give a determinate system.[5] The left-hand side of (3.2) may be called rate of preference between x and y (e.g., rate of time preference if x and y are amounts of milk to be consumed in two periods); it depends on tastes. The right-hand side may be called transformation rate between x and y, i.e., the rate at which x and y can be substituted for one another if assets and all other yields are kept constant: it depends on transformation (in Crusoe case, production) conditions. The two rates are equal "in equilibrium."

It will be noticed that the transformation condition, $T(x, y, \cdots; a, b, \cdots)=0$, describes the case of joint supply *and* joint demand. Each combination of m assets (=each balance sheet) can produce an n-dimensional set of yield combinations (=consumption plans). To select from this set the best consumption plan is, for the individual, "a matter of taste" as shown by the condition $U=\max$. If there is only one consumption good (and only one time period), the function T degenerates into the "production function" treated in the theory of distribution, wages, etc.: the case of joint supply only. If $a, b, \cdots$ (as in the Crusoe case) are fixed, T is identical with Irving Fisher's "opportunity function."[5a]

4. A MAN IN THE MARKET

Next, let us consider the case of a producer exchanging assets at give prices $p, q, \cdots$. The unknowns are: m asset quantities $a, b, \cdots$ and n yields $x, y, \cdots$.

The transformation condition $T(x, y, \cdots; a, b, \cdots)=0$ describes again all the consumption plans which are accessible to a present possessor of the amounts $a, b, \cdots$ of the assets, according to his judgement. The individual can realize these plans not only by future production (as in the Crusoe case) but also by future buying and selling of

[5] We shall not give the well-known discussion of second derivatives and the conditions under which a maximum, and not a minimum, is obtained.

[5a] R. G. D. Allen calls this rudimentary T a "transformation function" (*Mathematical Analysis for Economists*, pp. 121–124). He also treats the usual "production function" but he does not give a general function involving several production and several consumers' goods. Pareto (*Manuel*, App. 78) introduces a set of equations (in our notation): $a=F(x, y \cdots)$, $b=G(x, y \cdots)$; but these may become incompatible if there *are* more production goods than consumption goods.

assets. The transformation condition thus depends, not only on the expected technical conditions (including future inventions, weather, etc.) but also on the expected future market prices. We are thus concerned not only with "productive" but also with "speculative" stocks of goods and claims, i.e., those to be sold later.

Writing $a_0, b_0, \cdots$ for the initial quantities of assets (now to be distinguished from the unknown best amounts $a, b, \cdots$), we have

$$T(x, y, \cdots; a, b, \cdots) = 0, \text{ (transformation equation)}, \tag{4.1}$$

$$p(a - a_0) + q(b - b_0) + \cdots = 0, \text{ (balance equation)}, \tag{4.2}$$

and the requirement that $U(x, y, \cdots)$ should be a maximum. Differentiating,

$$\begin{aligned} U_x dx + U_y dy + \cdots &= 0 \\ T_x dx + T_y dy + \cdots + T_a da + T_a db &= 0 \\ pda + qdb + \cdots &= 0. \end{aligned}$$

Keeping the assets constant and letting only two yields vary at a time we get, as before, $n-1$ equations of the type

$$U_x/U_y = T_x/T_y, \text{ (yield-substitution equations)}. \tag{4.3}$$

Keeping, on the other hand, yields constant and varying two assets at a time, we get $m-1$ equations of the type

$$p/q = T_a/T_b, \text{ (asset-substitution equations)}. \tag{4.4}$$

We have thus $m+n$ equations for the same number of unknowns. The equation (4.4) shows that "in equilibrium" the price ratio between two assets is equal to their rate of transformation. It is more usual to state that "prices of production goods are proportionate to their marginal productivities" but the latter have seldom been defined so as to cover the case of joint demand and joint supply. Consider the derivative dT/da which is obtained from (4.1) if all yields but only one asset are regarded as variable:

$$\frac{dT}{da} = T_a + T_x \frac{dx}{da} + T_y \frac{dy}{da} + \cdots = 0;$$

similarly

$$\frac{dT}{db} = T_b + T_x \frac{dx}{db} + T_y \frac{dy}{db} + \cdots = 0.$$

Therefore, and in consequence of (4.3),

$$\frac{T_a}{T_b} = \frac{U_x \dfrac{dx}{da} + U_y \dfrac{dy}{da} + \cdots}{U_x \dfrac{dx}{db} + U_y \dfrac{dy}{db} + \cdots},$$

so that we can write, instead of (4.4), the following "marginal productivity theorem":

$$(4.4') \quad \frac{p}{q} = \frac{U_x \dfrac{dx}{da} + U_y \dfrac{dy}{da} + \cdots}{U_x \dfrac{dx}{db} + U_y \dfrac{dy}{db} + \cdots}, \text{ (marginal productivity equations).}$$

The numerator in the right-hand part of the last equation may be called for brevity the "marginal productivity" of the asset *a;* although a more precise name would be "the utility of the best accessible combination of marginal yields of the asset *a*"; or "the asset's largest possible contribution to the total utility of the consumption plan."[6]

Let x and y be the amounts of some definite item in the family budgets of two successive years; and suppose that the asset a contributes to nothing but the consumption of this item in the first year, while the asset b contributes the same amount to the consumption of the second year only. (These are, in fact, the assumptions tacitly made in the usual explanation of the interest rate as the "price of waiting.") We have then $dx/da = dy/db$; $dx/db = dy/da = 0$; while terms involving dz, etc., vanish. It follows from (4.4') that $p/q = U_x/U_y$, i.e., the price ratio is equal to the time-preference rate. The expression $(U_x/U_y - 1)$ is occasionally called the subjective rate of interest (for the two years concerned), and it must be equal, in equilibrium, to the market rate of interest $(p/q - 1)$. The assumptions made do not, however, apply in reality, except in the case of the lending of a consumption good, to be repaid *in one single lump.* In general there is no simple relationship between the market price of assets and the preference rates between yields (of which the preference, for a given kind of yield, between two given years is a special case). There is, therefore, strictly, no single "market rate of interest."

The usual assumptions are unrealistic also for the further reason that loans are seldom made in the form of consumption goods. The thing borrowed and lent is *cash.* Let, then, our asset a be "present cash," and our asset b be "a claim on next year's cash." The ratio p/q between the present prices of these two assets depends not only (as in the preceding paragraph) on the preference rates such as U_y/U_x but also on the comparative size of the marginal yields dx/da, dy/db; in the preceding paragraph these were assumed equal; but in the present meaning of a and b they depend on the money prices of consumption goods in the two years. This influence of the expected price tendency

[6] It is easily seen that Marshall's "Mathematical Appendix XIV" treats quite a different question.

of goods upon the present price of loans (or, loosely, upon the "market rate of interest") has been discussed in detail by Irving Fisher[7] and others.

Let a still be present cash; but let b be, not a present claim on future cash, but some present commodity stock. The numerator in (4.4′) is again the smaller, the higher the expected money prices of consumption goods: the well-known "flight from money into things" in times of expected inflation.[8]

The equation (4.4′) shows that the system may become incompatible (and therefore, in the economists' language, no equilibrium possible) if the preference rates U_x/U_y, etc., as well as the ratios $(dx/da)/(dx/db)$, etc., are constant. Preference rates are variable if there is some "law of changing (e.g., diminishing), marginal utilities"; while the ratios $(dx/da)/(dx/db)$, etc. are variable if there is some "law of changing (e.g., diminishing) returns." The latter must be generalized so as to include—in accordance with the definition of the transformation function—not only yields from production but also from speculation. The condition that $(dx/da)/(dx/db)$ should depend on a, b implies in this case that the investor must not assume the future market to be perfect; while the constancy of p/q assumed in this section implies a perfect present market. A speculator with a constant time-preference ("patience") rate would pile up "infinite" commodity stocks if he could buy spot and sell forward in perfect markets and was sure that future commodity prices will exceed present ones. In reality this limiting case is precluded not only by the imperfection of markets, both future and present, but also by uncertainty: limitations which we shall study later.

In the Crusoe case no utility was, of course, provided by holding cash. Under the assumptions of the present section cash holding receives some explanation in so far as cash may be held, even in conditions of certainty, to benefit from an expected *fall* of commodity prices. It has been shown how this affects the price of cash both in terms of commodities and of loans. This speculative aspect of cash holding is, however, obviously insufficient to explain it in general.

5. PRICE FORMATION IN A PERFECT MARKET

In the preceding section the asset prices $p, q, \cdots$ of the assets $a, b, \cdots$ were regarded as data given to the individual. Their formation can now be explained by the tastes, transformation conditions, and initial assets of all (say r) individuals in the market. The unknowns are:

[7] In the concluding part of his *Theory of Interest.*

[8] The "self-acceleration" of inflation follows from the above only if an additional, dynamic assumption holds true: that the expected price change has the same sign as the preceding one.

mr assets, nr yields, m prices; total, $mr+nr+m$. There are $(m+n)r$ equations of the types discussed in Section 4: r transformation equations (4.1), r balance equations (4.2), $r(n-1)$ yield-substitution equations (4.3), and either $r(m-1)$ asset-substitution equations (4.4) or $r(m-1)$ marginal-productivity equations (4.4′). In addition, there are m equations of the type

$$(5.1) \qquad \sum(a - a_0) = 0, \text{ etc., (clearing-of-the-market equations),}$$

the summation being performed over the r individuals. One of the equations of the type (5.1) is redundant because multiplying each by the corresponding price and adding we get the sum of all balance equations of the type (4.2). On the other hand, assuming a to be the numéraire, the equation

$$(5.2) \qquad p = 1$$

makes the necessary total, $mr+nr+m$ equations.

The contents of this section are well known but had to be briefly restated, as a link with the further discussion.

6. IMPERFECT PRESENT MARKET

The preceding sections did not exclude the imperfection (in the investor's eyes) of the *future* market: it was, indeed, indicated that it is more realistic to assume it, which simply meant that the transformation function $T(x, y, \cdots; a, b, \cdots)$ was nonlinear everywhere: imperfection of the future market (prices decreasing as sales increase) was treated as a case of "decreasing returns." If we had introduced the concept of "future assets" we could have distinguished formally between future exchange and future production; but this advantage would only have been gained at the expense of a much more complicated notation which I prefer to avoid at present. It suffices to remember always that the transformation function has been generalised so as to include under yields $x, y, \cdots$ all possible future consumption, whether made possible by future physical transformation or by future exchange.

The preceding sections did, however, assume the *present* market to be perfect. The rates of exchange between assets are, in a perfect market, independent of amounts exchanged. The linear "balance equation" (cf. 4.2)

$$(6.1) \quad pa + qb + \cdots - (pa_0 + qb_0 + \cdots) = 0, \text{ (balance equation),}$$

expresses the fact that each individual's receipts equal his expenditure in terms of any one asset chosen as a numéraire. In conditions of perfect competition, (6.1) can, however, also be interpreted as a "market

equation": the partial derivatives of its left-hand side taken at any point $(a, b, \cdots)$ with respect to a or b, etc., are proportionate to the respective prices. In conditions of imperfect competition, (6.1) remains valid as a balance equation; $p, q, \cdots$ being interpreted as the market prices actually paid for the total amounts bought—"average revenues." But it ceases to be valid as a "market equation" because the prices become dependent on the amounts $a, b, \cdots$; they become proportionate to functions, say, $p^*(a, b, c, \cdots)$, $q^*(a, b, c, \cdots)$ instead of $p, q, \cdots$. The market plane becomes a more general market surface. The individual has to choose not one out of the family of all "market planes" (6.1) through the initial point, but one of the family of "market surfaces" through the initial point, say

(6.2) $M(\mu, a, b, c, \cdots) - M(\mu, a_0, b_0, c_0, \cdots) = 0$, (market equation),

where $M = \int(p^*da + q^*db + \cdots)$, (assuming the right-hand side integrable). The r parameters μ (one for each individual) are new unknowns of the system which remains determined because r new equations (6.2) have been added. Thus, if the forms of the M-functions are given, the μ's (and, therefore, the relevant market surfaces for each individual) are determined by the tastes, initial assets, and transformation conditions of all individuals. The form of each M-function is determined by other data than those enumerated and is loosely described as the individual's "bargaining power," his "strategic position," etc. (In a perfect market, M degenerates into a linear form.) Substituting partial derivatives ("marginal revenues") M_a, $M_b, \cdots$ for $p, q, \cdots$ in the equations (4.4) or (4.4′) we get now

(6.3) $M_a/M_b = T_a/T_b$, (asset-substitution equations),

or

(6.3′) $$\frac{M_a}{M_b} = \frac{U_x \dfrac{dx}{da} + U_y \dfrac{dy}{da} + \cdots}{U_x \dfrac{dx}{db} + U_y \dfrac{dy}{db} + \cdots},$$ (marginal-productivity equations).

The equations (6.1) to (6.3) or (6.3′), together with (4.1), (5.1), and (5.2), constitute a determinate system.

For the problem of asset holdings, and in particular cash holdings, the imperfection of the present market is mainly relevant in connection with "decreasing selling costs": selling costs are expenses to improve the bargaining power (i.e., to straighten a market function); if they decrease, per unit of amount sold, with increasing sales, small sales will be avoided and stocks accumulated: e.g., stocks of cash

waiting for investment. Stocks (of cash or, say, wheat) due to the different timing of receipts and outgoings are cases of the same type.

7. UNCERTAINTY

Since, in the actual uncertain world, the future production situation (technique, weather, etc.) and future prices are not known, the transformation equation $T(a, b, \cdots; x, y, \cdots) = 0$ is not strictly valid so long as it means that, in the mind of the producer, to each combination of assets there corresponds one and only one n-dimensional set of yield combinations. It is more correct to assume (although this assumption will also be revised later) that to each combination of assets there corresponds, in his mind, an n-dimensional joint-frequency distribution of the yields. Thus, instead of assuming an individual who thinks he knows the future events we assume an individual who thinks he knows the probabilities of future events. We may call this situation the situation of a game of chance, and consider it as a better although still incomplete approximation to reality, and to relevant monetary problems, than the usual assumption that people believe themselves to be prophets.

We may, then, use the previous formal set-up if we reinterpret the notation: $x, y, \cdots$ shall mean, not future yields, but parameters (e.g., moments and joint moments) of the joint-frequency distribution of future yields. Thus, x may be interpreted as the mathematical expectation of first year's meat consumption, y may be its standard deviation, z may be the correlation coefficient between meat and salt consumption in a given year, t may be the third moment of milk consumption in second year, etc. We know of the economic relevance of certain of these parameters: e.g., in our illustration, x, z, and, in many cases, t are positive utilities, while y is a disutility: people usually like to eat more, rather than less, meat; they dislike (with the exception of passionate gamblers) situations in which the amounts of meat can be anything within a wide range; they like meat consumption to be accompanied by salt consumption; and (witness football pools) they like "long odds," i.e., high positive skewness of yields. It is sufficiently realistic, however, to confine ourselves, for each yield, to two parameters only: the mathematical expectation ("lucrativity") and the coefficient of variation ("risk"); while it would be definitely unrealistic (as pointed out in Keynes' *Treatise on Probability* as well as by Irving Fisher, A. C. Pigou, S. Florence, and J. R. Hicks), to confine ourselves to the mathematical expectation only, which is the usual but not justifiable practice of the traditional calculus of "moral probabilities."

The system of our Sections 3–6 holds good with this new interpretation. The unknowns are: the present quantities and prices of assets,

and the parameters of yield distributions. The data are: initial balance sheets, tastes of men (in terms of lucrativities and risks of various yields), and transformation conditions (expressed in terms of assets, and of the lucrativities and risks of various yields). As before, the rate of preference (or aversion) between x and y (say between risk and lucrativity of meat) equals, in equilibrium, their rate of transformation; and the price of each asset is proportionate to its marginal productivity, the latter being redefined as the utility of the best set of its marginal contributions to various parameters (e.g., to risk or lucrativity of meat, salt, etc.).

This application of the methods of the general price theory to the conditions of uncertainty is obviously relevant to monetary theory because of the relatively low risk which is usually attached to money. Cash always produces constant (namely zero) dividends; and, if it is disposed of, the consumption made thus possible depends on commodity prices only. Bonds also produce constant dividends; but the consumption made possible by selling a bond depends not only on commodity prices (in terms of cash) but also on loan prices (in terms of cash). Finally, with shares both prices and dividends vary. The marginal contribution of cash to the variability of future yields (i.e., consumption amounts) is therefore smaller than that of shares (and also plants, etc.) or even bonds (or houses, landed property, etc.).[9] A change in tastes in the sense of increasing risk aversion leads to decreasing prices of relatively risky assets, or to hoarding, or to both. The same result can be due to changes of transformation conditions in the sense of an increasing expected variability of yields (for given amounts of assets), i.e., to increased "insecurity feeling."

8. PLASTICITY, SALEABILITY, LIQUIDITY

Suppose the individual has fixed his present assets, as determined by the conditions described in the previous section. It can be shown that the range of variations of the *best* yields compatible with the assets thus fixed is the larger (1) the larger the dispersion of the frequency distribution of *possible* yields, (2) the greater the difficulty of substitution between the various yields, as measured by the curvature of the transformation function, (3) the larger the curvature of the indifference lines. We shall confine ourselves to a simple case of two kinds of yields only, say x and y.

[9] This is sometimes expressed by using the concept of a market rate of interest: the price of a share or bond is said to vary "owing" to fluctuations in the interest rate; (the price of a share varying also on account of fluctuations in dividends). Since the "market rate of interest" is only another word for relative asset prices (see Section 4), this is a tautology rather than an explanation.

Let the assets $a, b, \cdots$ assume their best present values. For yields, use the notation of the sections previous to 7. Express the random character of the transformation relationship between the yields x and y ($a, b, \cdots$ being fixed) not by using means and coefficients of variation, but by using a *random term* h, thus

$$y = f(x + h). \tag{8.1}$$

As h assumes various values with given probabilities, the best values of x and y, as required by the taste condition $U(x, y) = \max$, also change. They must satisfy the condition—cp. (3.2)—

$$\frac{d}{dx} f(x + h) = -\frac{U_x}{U_y} \cdot \tag{8.2}$$

Writing for brevity $-U_x/U_y = v(x, y)$, we have from (8.1) and (8.2)

$$v[x, f(x + h)] = f'(x + h).$$

Differentiating completely and separating the terms in dx and dh

$$dx = dh\left[-1 + \frac{\dfrac{\partial v}{\partial x}}{\dfrac{\partial v}{\partial x} + \dfrac{\partial v}{\partial f} f'(x + h) - f''(x + h)}\right]. \tag{8.3}$$

The fraction on the right-hand side has a positive numerator because $|v|$, the numerical value of the exchange ratio of x in terms of y, usually decreases as x increases. Since, on the other hand, $|v|$ increases with increasing y, and because of the law of diminishing returns, the denominator appears to be a sum of positive terms

$$\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial f} \cdot v\right) \qquad \text{and} \qquad [-f''(x + h)],$$

and is larger than the numerator. Hence dx/dh lies between -1 and 0, approaching -1 as the numerical value of $f''(x+h)$, i.e., the curvature of the transformation line, increases, or the sum of the two other terms of the denominator, i.e., the curvature of the indifference line, increases, or both.

If, therefore, the addition of a unit of some asset in forming a present balance sheet makes the curvature $f''(x)$ larger than the addition of a unit of some other asset, the risk contributed by the former is larger, and its marginal productivity (in the sense of Section 7), accordingly, smaller.

With a given dispersion of the transformation function (i.e., given

degree of "insecurity feeling")—measured, e.g., by the mean square root of the values of the random shifts (such as h), the price of the asset a is affected by the magnitude $d/da[f''(x)]$. Its reciprocal may be called "plasticity" of an asset: the easiness of manœuvring into and out of various yields after the asset has been acquired. When plasticity is due to the fact that the future market of this asset is perfect, we may call it "saleability": the more imperfect the future market of an asset the more difficult it is to sell it in order to adapt the yields consequent upon its acquisition to the transformation conditions as they take actual shape in the course of time.

The future market of cash is probably less imperfect than that of most assets. Cash is "universally acceptable" or has at least a very wide market. Cash is therefore held partly for the special purpose of future manœuvring. Holdings of cash and its price in terms of other assets must therefore be larger the larger the dispersion of the estimates of future transformation conditions.

Money's saleability, or plasticity, is probably meant when people use the word "liquid"—as opposed to "frozen." Yet the other property of cash already mentioned in Section 7—the low variability of its price, due to the low variability of its contribution to yields—is also important, and it may be useful to indicate the two properties by the single word "liquidity," admitting, however, that it denotes a bundle of two measurable properties and is therefore itself not measurable.

Future adjustment of yields can be performed by physical transformation of assets, as well as by selling and buying assets: saleability is not the only form of plasticity. Stocks of raw materials are more plastic than those of finished goods because they are more easily put into various physical shapes according to needs.[10]

9. DEGREE OF KNOWLEDGE

In the preceding two sections, the individual was assumed to know the relevant probabilities: the parameters of the transformation function as defined were assumed to be known. But this knowledge of probabilities—a situation approached in the games of chance—is a limiting case only. In reality the man does not regard himself as enabled by his experience to assign definite probabilities to each yield combination. Let the form of the transformation function (8.1) depend on a set of parameters $\boldsymbol{S}$. For each set of values, say, S_1 assigned to these parameters there can be stated a probability $P(\boldsymbol{E}|S_1)$ that the actual observed facts $\boldsymbol{E}$ (e.g., the crops and outputs, prices, etc., of

[10] To distinguish formally between the two kinds of plasticity would imply the treatment of "future assets" as variables of the system. We have found it unnecessary to use that more complicated approach.

the past, or any other information available) would have happened if S_1 was true. We obtain thus a likelihood function $L(S)$ of the variable set S, viz.,

$$L(S) = P(E \mid S).$$

If knowledge is small, the function does not show any conspicuous peaks: a set of parameters, S_1, may be nearly as much in agreement with the facts E as some other set, say, S_2 in its neighbourhood, i.e. $P(E \mid S_1)$ may be nearly equal to $P(E \mid S_2)$. Greater knowledge means a concentration of the most likely rival hypotheses within a narrow region. Characteristic parameters can be used to measure this: e.g., the steepness of $L(S)$ near its maximum, or some measure analogous to measures of dispersion, or Dr. Neyman's "confidence limits," etc.

This may be the *rationale* of Professor Knight's important distinction between "risk" and "uncertainty": the former is a known parameter of a frequency distribution, the latter, the lack of knowledge of this (or any other) parameter.[11]

The importance of this from the point of view of economic theory lies in the distinction between gambler and entrepreneur. Some assets may produce higher known risks than others, and gamblers may develop a higher preference for them than other people. On the other hand, some assets may contribute to the yields—their mean values as well as their variabilities—in a less-known way than others; and "entrepreneurs" have a higher preference for them than other people. Cash is usually regarded as an asset, the future productivity of which is definitely zero, and the future price of which (in terms of consumption goods) is assumed to remain known. Cash holdings grow whenever the prevailing preferences are nonentrepreneurial.

10. STATIC REWARDS AND DYNAMIC LOSSES

The above discussion has not included any statement as to whether the transformation functions expected by each individual have proved right or wrong in the next point of time after his "present" best collection of assets has been formed. Usually the forecasts will have proved more or less wrong; the total utility of each individual will have proved smaller than it would have been if he had known the future: these will be *dynamic losses*—(or malinvestment losses)—the larger, the less the man's skill (or luck) in forecasting. In what way dynamic losses influence the man's further revision of his expectations and thus

[11] We prefer to use "lack of knowledge" for Prof. Knight's "uncertainty" and to reserve the latter term for its ordinary use, viz., for expressing the fact that not all probabilities—even if known—equal 1 (and therefore not all dispersions equal 0).

affect the equilibrium of the day after tomorrow (as perhaps suggested by Mr. Keynes' phrase of "profits and losses being the mainspring of action"), is a problem in Economic Dynamics where the causation of changes of data (tastes, expectations, present market conditions) is to be studied.

Dynamic losses resulting from the change in data have clearly little to do with "static rewards" accruing to any individual whose tastes or expectations are different from those of the mass of other men. If meat abstinence is scarce, vegetarians profit by low prices of vegetables; a man who has a small time preference, or a small risk aversion, or optimistic ideas about future transformation, or an entrepreneurial taste for "dark horses" in times when these attitudes are scarce gets higher total utility than in times when they are common: the assets for which he is relatively more keen than others are cheaper, in terms of other assets, than in times when everyone develops those characteristics.

It follows from the preceding discussion that hoarding of cash is "rewarded" in those times. On the other hand, risk aversion, pessimistic transformation functions, and lack of either knowledge or entrepreneurial spirit, although analytically distinguishable, usually go, in fact, together, and all make for "cheap money."

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THE SIGNIFICANCE OF THE CHARACTERISTIC SOLUTIONS OF MIXED DIFFERENCE AND DIFFERENTIAL EQUATIONS

By R. W. James and M. H. Belz

1. Mixed difference and differential equations—functional relations involving the values of a function and its derivatives at different values of the argument—are beginning to assume considerable importance in econometrics, especially in connection with dynamical theories of the trade cycle.[1] Some progress has been made towards the solution of a simple equation of this type, to the extent that tables are now available for the calculation of the parameters which determine the periodicity and damping of the expocyclic characteristic solutions.[2] It has further been suggested, tentatively, that a solution of a difference-differential equation might be developed in terms of an infinite series of "characteristic" solutions. It is the purpose of this paper to investigate the conditions under which such a development is possible, and, in addition, to give methods for determining the coefficients of the development, when it exists.

It is further shown that the solutions of certain forms of integral, and integro-differential, equations can be given in the form of an infinite series derived from a consideration of related difference-differential equations.

2. The type of mixed difference and differential equation of most frequent occurrence is the first-order equation of the form

$$\theta\dot{y}(t) = ay(t) - cy(t - \dot{\theta}), \qquad (a, c, \theta \text{ real constants}) \tag{1}$$

in which there appears the "retarded" term, $cy(t-\theta)$. The selection of the unit of time is here arbitrary, and it will be found convenient, in the sequel, to choose θ as the unit. With this modification (1) becomes

$$\dot{y}(t) = ay(t) - cy(t - 1). \tag{2}$$

On putting $y(t) = Y(t)e^{at}$, this equation takes the form

$$\dot{Y}(t) = - ce^{-a}Y(t - 1). \tag{3}$$

From (3) it is seen that if $Y(t)$ is specified at all points in a unit range of t, say the range (0,1), then its derivative is specified at all

[1] Tinbergen, *Weltwirtschaftliches Archiv*, Vol. 34, 1931, p. 152; Econometrica, Vol. 3, 1935, p. 241; Kalecki, Econometrica, Vol. 3, 1935, p. 327; Frisch, *Economic Essays in Honour of Gustav Cassel*, p. 171.

[2] Frisch and Holme, Econometrica, Vol. 3, 1935, p. 225; James and Belz, Econometrica, Vol. 4, 1936, p. 157. The term "expocyclic" is here used to describe the product of an exponential and a sinoidal function.

points in the following range (1, 2). Thus, if t lies in the range (1, 2), we have, on writing $\tau = t-1$, so that τ lies in the range (0, 1),

$$\dot{Y}(t) = -ce^{-a}Y(\tau). \tag{4}$$

We shall require that $Y(t)$ is integrable in the range (0, 1). It is not necessary to discuss the broadest possible conditions for the integrability of $Y(t)$ in the Lebesgue or Riemann sense. It suffices here to note that a bounded function with a finite number of discontinuities in the range (0, 1) of t is integrable in that range. As all functions likely to be met with in the physical (or economic) world presumably satisfy this condition we need have little concern with the integrability conditions, although it might be borne in mind that the theory applies to a wider class of function than is encountered except in the realm of abstract analysis. Given the integrability of $Y(t)$ in the range (0, 1), we have, for the range (1, 2),

$$Y(t) = -ce^{-a}\int^{t-1} Y(\tau)d\tau + A, \qquad (A \text{ arbitrary}). \tag{5}$$

Moreover, the indefinite integral here is a continuous function of t, which is fixed once the constant of integration, A, is specified. This constant may be determined, for example, by prescribing the value of $Y(t)$ at one point in the range (1, 2). When $Y(t)$ is fixed, $y(t)$ is immediately determined from the relation $y(t) = Y(t)e^{at}$.

We can now apply the same procedure to determine $Y(t)$, or $y(t)$, in the range (2, 3), provided that its value at one point of the range is specified, and so on for the subsequent unit ranges. Thus it may be stated that if $Y(t)$, or $y(t)$, is specified as an integrable function in the range (0, 1) of t, it is determined for all subsequent instants, provided its values are given for at least one point in each of the unit ranges (1, 2), (2, 3), $\cdots$.

We are not, in effect, concerned with the values of $Y(t)$ prior to the instant $t=0$, but it is at once seen from (3) that if $Y(t)$ is differentiable in the range (0, 1), the value of $Y(t)$ is defined throughout the range $(-1, 0)$. If this function is differentiable in the range $(-1, 0)$, $Y(t)$ is defined throughout the range $(-2, -1)$; and so on.

Let us now return to the case where t is positive, and suppose that $y_0(t)$ is a given, integrable function of t. The function $y(t)$ may now be specified in the range (0, 1) by identifying it with $y_0(t)$ in the same range.[3] In what follows this function, $y_0(t)$, will be termed the *constraint*. On substituting $y_0(t)$ for $y(t)$ in the retarded term of (2), we obtain the solution (5), in terms of $y(t)$, in the form

[3] The function $y_0(t)$ need only be integrable in its range of definition (0, 1). It may, of course, exist and be integrable over a more extended range.

$$y(t) = A_1 e^{at} - ce^{at} \int^t y_0(t' - 1)e^{-at'} dt', \tag{6}$$

where A_1 is arbitrary. Since the relation $y(t) = y_0(t)$ is not necessarily valid outside the range (0, 1) of t, (6) gives $y(t)$ in the range (1, 2) but not necessarily elsewhere. Accordingly, it is convenient to denote the function on the right-hand side of (6), for any value of t, by the symbol $y_1(t)$, the suffix indicating that $y(t) = y_1(t)$ in the range (1, 2) but not necessarily elsewhere.

In a similar manner, writing

$$y_2(t) = A_2 e^{at} - ce^{at} \int^t y_1(t' - 1)e^{-at'} dt', \qquad (A_2 \text{ constant}), \tag{7}$$

we have $y(t) = y_2(t)$ in the range (2, 3) but not necessarily elsewhere.

Proceeding in this way, we define a set of functions $y_1(t)$, $y_2(t)$, $\cdots$ for all positive values of t, and the "solution" of the equation (2) is found as the function which is identified with the functions $y_0(t)$, $y_1(t)$, $\cdots$, $y_r(t)$, $\cdots$ in the respective unit ranges (0, 1), (1, 2), $\cdots$, $(r, r+1)$, $\cdots$ of t. The technique of "solving" a difference-differential equation is thus seen to lie in reducing it to a simple differential equation.

It is obvious that there is an infinite number of possible solutions arising from a given constraint, depending on the values assigned to the constants of integration, $A_1, A_2, \cdots$. There is one solution, however, which appears to be of special physical significance. Since the range functions, with the possible exception of the constraint, are continuous in their ranges of definition, being expressed in terms of indefinite integrals, the only possible points of discontinuity of $y(t)$, apart from those contained in the constraint, are at $t = 1, 2, \cdots$. It is possible to select the constants of integration so that the solution is continuous at these points, i.e., so that $y(r-0) = y_{r-1}(r) = y(r+0) = y_r(r)$ for $r = 1, 2, \cdots$. The relation $y_{r-1}(r) = y_r(r)$, for $r = 1, 2, \cdots$, is termed the *condition of continuity*, and defines the *continuous solution* arising from a given constraint.

This unique continuous solution would appear to correspond to the physical reality of the problem. If a physical system is "left to itself" we should not expect to find abrupt discontinuities. Discontinuities only appear to arise in economic mechanisms due to an outside influence independent of the mechanism itself. If we suppose that no outside influences operate after the constraint has "started off" the system, the continuous solution would naturally be the one to take.

As an example of the determination of the continuous solution, let us consider the case corresponding to the constraint $y_0(t) = 0$ in the range (0, 1). From (6) we ascertain that, for the range (1, 2),

$$y_1(t) = A_1 e^{at}. \tag{8}$$

From the condition of continuity we have that $y_1(1) = y_0(1) = 0$. But from (8), $A_1 = y_1(1)e^{-a} = 0$. Therefore, $y_1(t) = 0$. In a similar manner it can be shown that $y_2(t) = y_3(t) = \cdots = 0$. The solution of (2) corresponding to the constraint $y_0(t) = 0$ is therefore $y(t) = 0$.

As a second example, take the case where the constraint has the constant value 1. Then

$$y_1(t) = A_1 e^{at} - ce^{at} \int^t e^{-at'} dt' = A_1 e^{at} + c/a. \tag{9}$$

Applying the condition of continuity at $t = 1$, we have $y_1(1) = 1$, and hence

$$A_1 = (1 - c/a)e^{-a}. \tag{10}$$

Again, we have

$$\begin{aligned} y_2(t) &= A_2 e^{at} - ce^{at} \int^t \{A_1 e^{a(t'-1)} + c/a\} e^{-at'} dt' \\ &= A_2 e^{at} - A_1 cte^{a(t-1)} + c^2/a^2. \end{aligned} \tag{11}$$

The condition of continuity at $t = 2$ gives

$$A_2 = A_1\{1 + (2c + c/a)e^{-a}\}.$$

Continuing in this way, $A_3, A_4, \cdots$ can be calculated in turn. The required solution is thus the function which is equal to 1 in the range (0, 1), to $c/a + (1 - c/a)e^{a(t-1)}$ in the range (1, 2), to $c^2/a^2 - (1 - c/a)cte^{a(t-2)} + (1 - c/a)\{1 + (2c + c/a)e^{-a}\}e^{a(t-1)}$ in (2, 3), and so on.

3. There are certain analytic functions of t which satisfy the equation (2) for all values of t. These are termed *characteristic solutions* of the equation. Thus, if $y_0(t)$ has the form of a characteristic solution, $y_1(t), y_2(t), \cdots$ will be identical with $y_0(t)$. This property may be termed the *self-perpetuating property* of a characteristic solution. We have seen that $y(t) = 0$ satisfies (2) for all values of t when $y_0(t) = 0$, so that it is a characteristic solution, and the constraint 0 may be described as a *self-perpetuating function.*

Characteristic solutions of (2) have been discussed by Frisch and Holme, and by James and Belz in the papers already mentioned. Non-oscillatory characteristic solutions occur when $c > 0$, $a - \log_e c \geqq 1$, and when $c < 0$. In the first case, if $a - \log_e c > 1$, there is a solution of the form $Ae^{\alpha t} + Be^{\beta t}$, where A, B are constants, and α, β are the (real) roots of the equation $a - v = ce^{-v}$; while if $a - \log_e c = 1$, there is a solution of the form $(A + Bt)e^{(a-1)t}$. When $c < 0$, there is always a solution of the form $Ae^{\alpha t}$, where α is now the (sole) real root of the equation $a - v = ce^{-v}$.

Expocyclic solutions exist for all values of a and c. In order to investigate them, we assume the form

$$y(t) = ke^{\rho t} + \bar{k}e^{\bar{\rho}t}, \tag{12}$$

where $\rho = v + iu$, k is a complex constant, and $\bar{\rho}$, $\bar{k}$ are the complex conjugates of ρ, k, respectively. On making this substitution in (2), we find, as the equation to be satisfied by ρ,

$$\rho = a - ce^{-\rho}. \tag{13}$$

There is an infinite number of solutions of this equation, termed the complex *characteristic numbers,* one corresponding to each of the ranges $(2n\pi,\ 2n\pi + 2\pi)$ of u, $(n = 1, 2, \cdots)$, while a solution corresponding to the value 0 of n exists when c is negative, or when c is positive and $a - \log_e c < 1$. The solution of (2) corresponding to the nth characteristic number is termed the nth *overtone,* while an expocyclic solution of (2) with a characteristic number having its imaginary part in the range $(0, 2\pi)$ may be termed the *undertone,* or long-wave solution, on account of the fact that its period is greater than unity.

We have spoken only of characteristic numbers for which u lies in the range $(2n\pi,\ 2n\pi + 2\pi)$, where n is a *positive* integer. Actually, of course, corresponding to the nth characteristic number, there is a number when u lies in the range $(-2n\pi - 2\pi,\ -2n\pi)$, which we may denote by ρ_{-n}. It is not difficult to see that $\rho_{-n} = \bar{\rho}_n$, i.e., that the characteristic numbers are *conjugate.* We are justified in pairing conjugate solutions, so that there is no occasion to speak of a $-n$th overtone. The characteristic number of an undertone solution, when one exists, has also a complex conjugate.

Generally speaking, the expocyclic solutions of (2) are damped, that is, the v's are negative. We have, in fact,[4]

$$v_n = -\log_e \left(\frac{u_n}{c \sin u_n}\right), \tag{14}$$

from which it is inferred that v_n is negative if $|u_n/\sin u_n| > |c|$. This inequality is clearly satisfied if $|c| < 2n\pi$. It follows at once that, if the nth overtone is damped, so are all overtones of order higher than n.

To find an asymptotic expression for the nth characteristic number when n is large, we notice from (14) that $-v_n$ tends to infinity with u_n, but not so rapidly. The relation[5]

[4] Compare Frisch and Holme, *loc. cit.*, equation (8), or James and Belz, *loc. cit.*, equation (5).

[5] Compare Frisch and Holme, *loc. cit.*, equation (9), or James and Belz, *loc. cit.*, equation (4).

(15) $$(v_n - a)e^{v_n} = - c \cos u_n$$

shows, since $v_n e^{v_n} \to 0$ as $v_n \to -\infty$, that $\cos u_n \to 0$, so that $\sin u_n \to \pm 1$. The sign of $\sin u_n$ must be the same as that of c. Hence we have, for large values of n, the approximate expressions,

(16) $$\rho_n \approx - \log_e \left\{(2n + \tfrac{1}{2})\pi/c\right\} + (2n + \tfrac{1}{2})\pi i$$

when c is positive, and

(17) $$\rho_n \approx - \log_e \left\{(2n + \tfrac{3}{2})\pi/\,|\,c\,|\,\right\} + (2n + \tfrac{3}{2})\pi i$$

for c negative.

The overtones are analogous to harmonics in Acoustics, but their frequencies bear no simple integral relationships to that of the "fundamental." The first overtone might, perhaps, be termed the fundamental of the system, since its period approximates most closely to unity, but there is no special feature differentiating it from the higher overtones, so that there is no reason for any terminological distinction.

The most general linear combination of characteristic solutions of (2) is of the form

(18) $$F_0(t) + \sum_{n=1}^{\infty} (k_n e^{\rho_n t} + \bar{k}_n e^{\bar{\rho}_n t}),$$

where the infinite series embraces all the overtones and $F_0(t)$ has the form

(i) $Ae^{\alpha t} + Be^{\beta t}$ when $c > 0$, $a - \log_e c > 1$, $(\alpha > 0 > \beta)$,

(ii) $(A + Bt)e^{(a-1)t}$ when $c > 0$, $a - \log_e c = 1$,

(iii) $k_0 e^{\rho_0 t} + \bar{k}_0 e^{\bar{\rho}_0 t}$ when $c > 0$, $a - \log_e c < 1$,

(iv) $Ae^{\alpha t} + k_0 e^{\rho_0 t} + \bar{k}_0 e^{\bar{\rho}_0 t}$ when $c < 0$.

When $F_0(t)$ has the form (iii), the expression (18) may be written

$$\sum_{0}^{\infty} (k_n e^{\rho_n t} + \bar{k}_n e^{\bar{\rho}_n t}),$$

which may be given in the form

$$\sum_{-\infty}^{\infty} (k_n e^{\rho_n t}).$$

It has been demonstrated that the fundamental property of a characteristic solution is that it is self-perpetuating. It follows from the linearity of (2) that a finite linear combination of characteristic terms is also self-perpetuating. Thus, if

(19) $$y_0(t) = F_0(t) + \sum_{1}^{N} (k_n e^{\rho_n t} + \bar{k}_n e^{\bar{\rho}_n t}),$$

$y(t)$ will be represented for all values of t by the same series.

The question naturally arises whether the self-perpetuating property is preserved when the number of terms in the series (19) is increased indefinitely, so that we get the series (18). If the infinite series is convergent in the range $(0, \infty)$, and has a derived series convergent in the range $(1, \infty)$, it defines a function which satisfies (2) in the range $(1, \infty)$. Under these conditions an infinite series of characteristic terms will be self-perpetuating. The series might, of course, converge also for some negative values of t.

In general, the constraint will not be given explicitly by means of a series of characteristic terms; $y_0(t)$ might, for example, be given as an algebraic function. The question then arises, whether such a constraint is converged to by an infinite series having, as a whole, the self-perpetuating property. This problem forms the subject matter of the next two Sections. It will be proved that, subject to the constraint being bounded and integrable, it is possible to represent the continuous solution by an infinite series of characteristic terms.

4. It is the purpose of the present Section to show that the continuous solution arising from a bounded constraint cannot tend to infinity more rapidly than a certain exponential, and further, that by the transformation $v(t)=y(t)e^{-\sigma t}$, we arrive at another difference-differential equation, the continuous solution of which tends to zero as $t\to\infty$.

From the equation (3) we arrive at the inequality

$$(20)\qquad \begin{aligned} &\int_r^t \left| \dot{Y}_r(t') \right| dt' \leqq |c|\, e^{-a} \int_{r-1}^{t-1} \left| Y_{r-1}(t') \right| dt', \qquad \text{or,}\\ &\left| Y_r(t) \right| \leqq \left| Y_r(r) \right| + |c|\, e^{-a} \left| Y_{r-1}(t-1) \right|_{\text{max.}}, \end{aligned}$$

where $|Y_{r-1}(t-1)|_{\text{max.}}$ is the maximum value of the modulus of $Y(t)$ in the range $(r-1, r)$ of t. As $|Y_r(r)| = |Y_{r-1}(r)| \leqq |Y_{r-1}(t-1)|_{\text{max.}}$, we have

$$(21)\qquad \left| Y_r(t) \right| \leqq (1 + |c|\, e^{-a}) \left| Y_{r-1}(t-1) \right|_{\text{max.}}.$$

This relation is true for all values of t in the range $(r, r+1)$; it must, therefore, be true for that value of t which makes $|Y_r(t)|$ a maximum in that range. Hence

$$(22)\qquad \begin{aligned} \left| Y_r(t) \right|_{\text{max.}} &\leqq (1 + |c|\, e^{-a}) \left| Y_{r-1}(t-1) \right|_{\text{max.}}\\ &\leqq (1 + |c|\, e^{-a})^{r-1} . \left(\left| Y_0(1) \right| + |c|\, e^{-a} \int_0^{t-r} \left| Y_0(t') \right| dt' \right). \end{aligned}$$

Hence, since $|Y_0(1)|$ is finite and $\int_0^1 |Y_0(t')|\, dt'$ exists, we have

$$\left| Y(t) \right| \leqq Ce^{\alpha t},$$

for all values of t, where C is a finite constant and $\alpha = \log_e(1 + |c|\, e^{-a})$.

It follows from this that

$$|y(t)| \leqq Ce^{(a+\alpha)t}.$$

Let us now introduce a new variable, $v(t)=y(t)e^{-gt}$, where g is assumed to be positive. It may at once be verified that if $y(t)$ is a function satisfying the relation (2), $v(t)$ satisfies the equation

$$\dot{v}(t) = (a - g)v(t) - ce^{-g}v(t-1), \tag{23}$$

where we have $|v(t)| \leqq Ce^{(a+\alpha-g)t}$. We have thus only to select a value of $g>a+\alpha$ in order to define a function which certainly tends to zero as $t\to\infty$, and for which, moreover, the integral $\int_0^\infty |v(t)|\,dt$ exists.

If it can be shown that a solution $v(t)$, arising from a constraint $y_0(t)e^{-gt}$, can be expanded in an infinite series

$$v(t) = F_0(t)e^{-gt} + \sum_1^\infty (k_n e^{\rho'_n t} + \bar{k}_n e^{\bar{\rho}'_n t}),$$

where the ρ''s are the characteristic numbers of the equation

$$\rho' = a - g - ce^{-(g+\rho')}, \tag{24}$$

then it follows at once that

$$y(t) = F_0(t) + \sum_1^\infty (k_n e^{\rho_n t} + \bar{k}_n e^{\bar{\rho}_n t}),$$

where the ρ's are now the characteristic numbers of (13). This is evident from the fact that the characteristic numbers of (13) and (24) are connected by the relation $\rho_n=\rho_n'+g$.

5. We have seen that $|v(t)| \leqq Ce^{-\mu t}$, where $\mu\ (=g-a-\alpha)$ is positive. It follows that $\int_0^\infty v(t)dt$ is an absolutely convergent integral, and consequently $v(t)$ may be expressed in the Fourier integral form,[6]

$$v(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ipt}dp \int_0^\infty e^{-ip\tau}v(\tau)d\tau. \tag{25}$$

This relation is valid for all positive values of t.

To arrive at an expression for $\int_0^\infty v(\tau)e^{-ip\tau}d\tau$, we consider the integral

$$I(R) = \int_0^{R+1} v(\tau)e^{-ip\tau}d\tau = \sum_{r=0}^{R}\int_r^{r+1} v_r(\tau)e^{-ip\tau}d\tau. \tag{26}$$

Making the substitution $v_r(\tau)=\dfrac{1}{a'}\dot{v}_r(\tau)+\dfrac{c'}{a'}v_{r-1}(\tau-1)$, where $a'=a-g$, $c'=ce^{-g}$, we find the relation

$$\begin{aligned}(a' - ip - c'e^{-ip})I(R) = {}&(a' - ip)\textstyle\int_0^1 v_0(\tau)e^{-ip\tau}d\tau - v_0(1)e^{-ip}\\ &+ v_{R+1}(R+1)e^{-ip(R+1)} - c'e^{-ip}\textstyle\int_R^{R+1}v_R(\tau)e^{-ip\tau}d\tau.\end{aligned} \tag{27}$$

[6] At a point of discontinuity of $v(t)$, which may be allowed in the range of the constraint, the left-hand side of equation (25) will be $\frac{1}{2}\{v(t+0)+v(t-0)\}$.

Since $v_R(\tau)\to 0$ as $R\to\infty$, we have

$$\int_0^\infty v(\tau)e^{-ip\tau}d\tau = \lim_{R\to\infty} I(R) \tag{28}$$

$$= \frac{(a'-ip)\int_0^1 v_0(\tau)e^{-ip\tau}d\tau - v_0(1)e^{-ip}}{a'-ip-c'e^{-ip}}.$$

Therefore, (25) may be written

$$2\pi v(t) = \int_{-\infty}^{\infty} \frac{(a'-ip)\int_0^1 v_0(\tau)e^{-ip\tau}d\tau - v_0(1)e^{-ip}}{a'-ip-c'e^{-ip}} e^{ipt}dp. \tag{29}$$

To determine the value of this integral, consider the complex integral

$$\int_{(C)} \frac{(a'-ip)\int_0^1 v_0(\tau)e^{-ip\tau}d\tau - v_0(1)e^{-ip}}{a'-ip-c'e^{-ip}} e^{ipt}dp, \tag{30}$$

taken in a counterclockwise sense around the rectangular contour whose corners are the points A, B, for which $p=\pm(4n+3)\pi/2$, and D, C, for which $p=\pm(4n+3)\pi/2+in$. We here suppose $c'>0$, the contour in this case being such that, when n is large, it does not pass through a zero of the function[7]

$$X(p) = a' - ip - c'e^{-ip}, \tag{31}$$

and, further, such that it contains all the zeros of $X(p)$ for which $|\xi|<(4n+3)\pi/2$, where $p=\xi+i\eta$. Then

$$\int_{(C)} = \int_A^B + \int_B^C + \int_C^D + \int_D^A.$$

If we assume that $v_0(\tau)$ is piece-wise differentiable, with a bounded derivative, for $0\leqq\tau\leqq 1$, it is not difficult to show that the modulus of the numerator in the integrand of (30) is less than $K_1e^{-(t-1)\eta}$, where K_1 is a constant.

Along BC we have

$$|a'-ip-c'e^{-ip}| > (4n+3)\pi/2 + c'e^\eta > K_2(n+e^\eta),$$

(K_2 constant),

and hence

$$\left|\int_B^C\right| < \int_0^n K\frac{e^{-(t-1)\eta}}{n+e^\eta}d\eta, \qquad (K \text{ constant}),$$

$$= \int_0^n K\frac{e^{-t\eta}}{1+ne^{-\eta}}d\eta = \int_0^{\frac{1}{2}\log_e n} K\frac{e^{-t\eta}}{1+ne^{-\eta}}d\eta$$

[7] The asymptotic expression for the zeros in this case is obtained from (16). The case $c'<0$ is treated similarly.

$$+\int_{\frac{1}{2}\log_e n}^{n} K\frac{e^{-t\eta}}{1+ne^{-\eta}}\,d\eta$$

$$<\int_{0}^{\frac{1}{2}\log_e n} \frac{K}{1+\sqrt{n}}\,d\eta+\int_{\frac{1}{2}\log_e n}^{n} Ke^{-t\eta}d\eta, \qquad (t>0),$$

$$=\tfrac{1}{2}K\,\frac{\log_e n}{1+\sqrt{n}}+\frac{K}{t}\,(n^{-t/2}-e^{-nt}).$$

It follows that $\left|\int_B^C\right|\to 0$ as $n\to\infty$.

Similarly, $\left|\int_D^A\right|\to 0$ as $n\to\infty$.

Along CD we have $|a'-ip-c'e^{-ip}|>n+c'e^n$, and hence

$$\left|\int_C^D\right|<\int_{-(4n+3)\pi/2}^{(4n+3)\pi/2} K\,\frac{e^{-(t-1)n}}{n+c'e^n}\,d\xi=\frac{Ke^{-nt}}{c'+ne^{-n}}\,(4n+3)\pi$$

$$<K^*e^{-nt}(4n+3)\pi, \qquad (K^* \text{ constant}).$$

Therefore, for $t>0$, $\left|\int_C^D\right|\to 0$ as $n\to\infty$.

We conclude that, in the limit as $n\to\infty$, the integral (30) is equal to the integral on the right-hand side of (29), provided that $t>0$.

But by Cauchy's residue theorem this complex integral is equal to $2\pi i$ times the total number of residues arising from poles enclosed by the path of integration. These poles are infinite in number, and occur at the zeros of $X(p)$, i.e., for $ip=\rho_n'$, where ρ_n' is a characteristic number of (24). The function $v(t)$ has been defined in such a way that all the characteristic numbers of (24) have negative real parts, so that all the poles of the integrand are enclosed by the contour taken. It is not difficult to show, in addition, that all the poles are simple, except in the case where $a-\log_e c=1$.

In the case of a simple pole at $ip=\rho_n'$, the value of the residue is given by the expression

$$\frac{(a'-\rho_n')\int_0^1 v_0(\tau)e^{-\rho_n'\tau}d\tau-v_0(1)e^{-\rho_n'}}{\left[\dfrac{d}{dp}X(p)\right]_{ip=\rho_n'}}\,e^{\rho_n't}$$

(32)

$$=\frac{(a'-\rho_n')\int_0^1 v_0(\tau)e^{-\rho_n'\tau}d\tau-v_0(1)e^{-\rho_n'}}{i(c'e^{-\rho_n'}-1)}\,e^{\rho_n't}.$$

Summing for all these poles we obtain the relation

$$(33) \qquad v(t) = \sum_{-\infty}^{\infty} (k_n e^{\rho_n' t}),$$

expressing the solution $v(t)$, resulting from a given constraint, in terms of an infinite series of characteristic terms, the coefficients being determined from the relation

$$(34) \qquad k_n = \frac{(a' - \rho_n')\int_0^1 v_0(\tau)e^{-\rho_n'\tau}d\tau - v_0(1)e^{-\rho_n'}}{c'e^{-\rho_n'} - 1}.$$

Since $\rho_{-n}' = \bar{\rho}_n'$, it may be verified that $k_{-n} = \bar{k}_n$, when $v_0(t)$ is real, as has been assumed all along. Hence it is possible to write the series (33) in the form (18) already given, for it must be pointed out that the expression (33) contains terms of the type, $Ae^{\alpha t}$, or $k_0 e^{\rho_0' t} + \bar{k}_0 e^{\bar{\rho}_0' t}$, which have so far been included in the term $F_0(t)e^{-gt}$.

By definition $y(t) = v(t)e^{gt}$, so that, as we have shown in the previous Section, $y(t)$ can be represented by a series of the form (18), where the k_n are given by expressions of the type (34), which is equivalent to

$$(35) \qquad k_n = \frac{(a - \rho_n)\int_0^1 y_0(t)e^{-\rho_n t}dt - y_0(1)e^{-\rho_n}}{ce^{-\rho_n} - 1}.$$

Hence, the solution of any equation of the form (2) *arising from a bounded and integrable constraint may be represented by an infinite series of characteristic terms, such a series representing the continuous solution for all positive values of t.*

A special consideration of the case where $a - \log_e c = 1$ is necessary on account of the fact that in this case $X(p)$ has a double zero at the point $ip = \rho_0 = a - 1$. The residue in this case, determined by standard methods, is found to be

$$\frac{2it}{c} e^{(a-1)t}\left[c\int_0^1 y_0(t)e^{(1-a)t}dt - y_0(1)\right]$$

$$+ \frac{2ie^{(a-1)t}}{c}\left[\tfrac{2}{3}y_0(1) - \tfrac{2}{3}e^{(a-1)}\int_0^1 y_0(t)e^{(1-a)t}dt - c\int_0^1 ty_0(t)e^{(1-a)t}dt\right],$$

so that $y(t)$ is now given by an equation of the form

$$y(t) = (A + Bt)e^{\alpha t} + \sum_{1}^{\infty} (k_n e^{\rho_n t} + \bar{k}_n e^{\bar{\rho}_n t}), \quad (\alpha = a - 1),$$

which, we have seen, is the most general form of a series of characteristic terms when $a - \log_e c = 1$.

6. The above demonstration of the developability of the continuous solution in a series of characteristic terms is based on the hypothesis

that the constraint is integrable and that $y_0(1)$ is finite. But in Section 2 it was shown that only under these conditions may there properly be said to be a solution. We thus conclude that when a solution of (2) exists it is representable by a series of damped terms.

It is of interest to see under what conditions some sort of solution exists when t is negative. From the asymptotic expression for ρ_n when n is large we know that $|e^{\rho_n t}| \approx 1/(2\pi n)^t$. The series $\sum_1^\infty k_n/n^t$ will be convergent for all $t > -\alpha$, if k_n is of the order $1/n^{1+\alpha}$, so that we may extend the range within which the series may be applied for certain types of constraint, the nature of which need not be discussed here.

With such constraints it is possible to develop a solution in terms of an infinite series of characteristic terms, even when the characteristic solutions are explosive. To investigate a typical case, consider the equation (2) under the transformation $t' = 1 - t$. In this way the equation

$$-\dot{y}(t'-1) = ay(t'-1) - cy(t') \tag{36}$$

is derived. With the substitution $y(t') = ke^{\rho t'} + \bar{k}e^{\bar{\rho} t'}$, the equation giving the characteristic numbers is

$$a + \rho = ce^\rho. \tag{37}$$

This is equation (13) with the sign of ρ reversed. Thus the characteristic frequencies are the same for (36) as for (2), but the sign of the damping exponent is reversed, giving progressively anti-damped overtones. If, as above, we have a constraint such that k_n is $O(1/n^{1+\alpha})$, a solution $y(t')$ may be developed in a series $F_0(t') + \sum_1^\infty (k_n e^{\rho_n t'} + \bar{k}_n e^{\bar{\rho}_n t'})$, even though $e^{\rho_n t'}$ is "explosive," the development being valid for $t' < \alpha$. Equations of the type (36) may appear in economic analysis, but their application is considerably less widespread than those having progressively damped characteristic solutions.

7. It is of interest to see how closely a few terms of a damped Fourier series actually fit a given solution. For this purpose take the equation

$$\dot{y}(t) = 0.9533y(t) - 1.023y(t-1) \tag{38}$$

subject to the constraint $y_0(t) = 1$ in the range (0, 1) of t. The above choice of parameters is somewhat arbitrary, but was made to correspond to Kalecki's case of an undamped long-wave solution of period 10 years.

The solution of (38) may be written in the form

$$y(t) = \sum_0^\infty (k_n e^{\rho_n t} + \bar{k}_n e^{\bar{\rho}_n t}), \tag{39}$$

where $$k_n = \frac{c\int_0^1 y_0(t)e^{-\rho_n t}dt - y_0(1)}{c - e^{\rho_n}} = \frac{(c-a)}{(c-e^{\rho_n})}\cdot\frac{1}{\rho_n}.$$

For the long-wave solution we find $\rho_0=0.3726i$, while for the first and second overtones, $\rho_1=-1.91+2.38\ \pi i$, $\rho_2=-2.63+4.42\ \pi i$.

To see how closely the finite series, $\sum_0^{N-1}(k_n e^{\rho_n t}+\bar{k}_n e^{\bar{\rho}_n t})$ approximates to the exact solution (39) it is only necessary to notice that the right-hand side of (39) is equal to the sum of the above finite series and the infinite series $\sum_N^\infty(k_n e^{\rho_n t}+\bar{k}_n e^{\bar{\rho}_n t})$, which is certainly less than the series $2\sum_N^\infty|k_n e^{\rho_n t}|$. But this series is less than the expression

$$\frac{c-a}{c-e^{v_N}}\cdot\frac{c^t}{(2\pi)^{t+1}}\cdot\sum_N^\infty\frac{1}{n^{t+1}}.$$

To determine the fit of the undertone to the exact solution (39), put $N=1$. When $t=1$ this expression is of the order 0.007. The actual de-

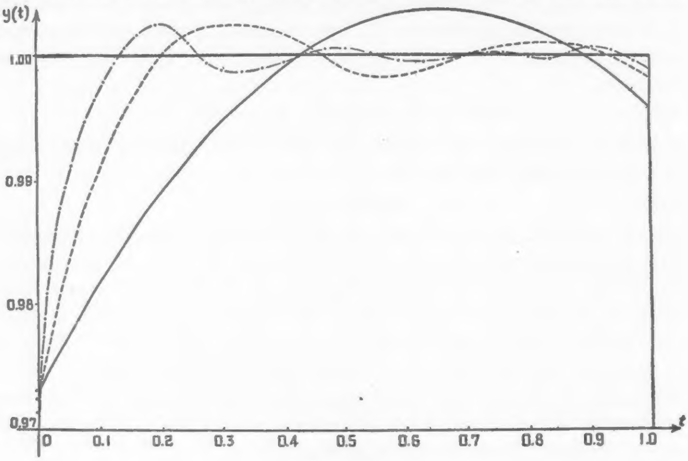

FIGURE 1

parture, as shown in Figure 1, is about 0.004. At $t=2$ the maximum possible departure is only about a tenth as large as for $t=1$, while in the time taken for a complete cycle of the undertone the difference between the undertone and the exact solution is reduced to a figure of the order 10^{-16}. Thus, after a very short time the undertone approximates as closely to the exact solution as could be desired, and this no matter what the form of the initial constraint. When the first overtone is also taken into account a still better approximation is obtained. At $t=1$ the maximum possible departure is of the order 0.002, while after a complete period of the long-wave solution the difference is of the order 10^{-21} only.

The fit of a finite series of characteristic terms to the exact solution is not so close in the range (0, 1), but even here, as Figure 1 shows, the

agreement is very good. The continuous curve represents the undertone only. In the uniformly broken curve the first overtone is added, and in the third curve the second overtone is taken into consideration as well.

It is evident that, except in the vicinity of $t=0$, the approximation to the exact solution is very close, and improves with each additional overtone. It is obvious from inspection that, as the number of terms taken increases, the approximation to the constraint becomes worse at $t=0$. In general the series does not converge to the constraint at this point. Another point of interest is that the series appears to approach the constraint more closely in the middle of the range than at $t=1$. This is to be expected from the fact that the derivative of $y(t)$ is discontinuous at this point.

In numerical terms the solution (39) may be written

$$\begin{aligned} y(t) = {} & 0.9732 \cos 0.3726t + 0.2463 \sin 0.3726t \\ & - e^{-1.91t}\{0.0015 \cos 2.38\pi t - 0.0160 \sin 2.38\pi t\} \\ & - e^{-2.63t}\{0.0011 \cos 4.42\pi t - 0.0097 \sin 4.42\pi t\} - \text{etc.} \end{aligned}$$

8. From a knowledge of the solution of equation (2) it is possible to give a series solution of the integral equation

$$x(t) = h \int_{t-1}^{t} x(t')dt', \tag{40}$$

for, by making the substitution $x(t)=\dot{y}(t)$, (40) may be written

$$\dot{y}(t) = h\{y(t) - y(t-1)\}. \tag{41}$$

Consider the series solution of (41),

$$y(t) = F_0(t) + \sum_1^\infty (k_n e^{\rho_n t} + \bar{k}_n e^{\bar{\rho}_n t}), \tag{42}$$

arising from some constraint. If this series is uniformly convergent in the range $(0, \infty)$, the series for $y(t-1)$ is uniformly convergent in the range $(1, \infty)$, and hence the series for $y(t)-y(t-1)$ is uniformly convergent in the range $(1, \infty)$. It follows from (41) that the series for $\dot{y}(t)$ is uniformly convergent in the same range, being given by

$$\dot{y}(t) = F_0'(t) + \sum_1^\infty (k_n \rho_n e^{\rho_n t} + \bar{k}_n \bar{\rho}_n e^{\bar{\rho}_n t}). \tag{43}$$

Thus, if the series (42) converges uniformly to the function $y(t)$ for all positive t, the derived series converges uniformly to the derivative of $y(t)$ for $t>1$.

The coefficients of the series (42) are given by relations of the type

$$(44) \qquad k_n(h - e^{\rho_n}) = h\int_0^1 y_0(t)e^{-\rho_n t}dt - y_0(1)$$

$$= \frac{h}{\rho_n}\left[\int_1^2 \dot{y}_1(t)e^{-\rho_n t}dt - e^{-\rho_n}\int_1^2 \dot{y}_1(t)dt\right].$$

Therefore the coefficients of the series (43) are found from

$$(45) \qquad k_n\rho_n = \frac{h\int_1^2 x_1(t)e^{-\rho_n t}dt - x_1(2)e^{-\rho_n}}{h - e^{\rho_n}}.$$

According to the above argument the convergence of the series (43) has only been demonstrated for the range $(1, \infty)$ of t, but for special types of constraint the series may be extended to the range (0, 1), and even for negative values of t.

It is worth noting in passing that there is always one more solution of (41) than of (40). The substitution $x(t)=A$ only satisfies (40) when $h=1$, whereas there is always a constant solution of (41). When $h=1$, we also have the solution Bt of (41), which has no counterpart in (40). It is in consequence of the set of characteristic solutions being "incomplete" that we are unable, in the general case, to represent a solution of (40) by a damped Fourier series for *all* positive values of t.

Some indication of the way in which an equation of the type (40) is expected to enter into mathematical economics might be given by assuming the variable $x(t)$ to represent the rate of new production of a commodity at time t. If the "period of gestation" of the commodity is unity, the total volume of goods in process of production at time t is given by $\int_{t-1}^{t} x(t')dt'$. Thus, the equation (40) expresses a proportionality between the rate of fresh production started at a given time and the volume of goods in process at that time, which may be regarded as an index of industrial activity. We might also imagine such a relation to hold, but with a lag. The equation determining $x(t)$ is then

$$(46) \qquad x(t) = h\int_{t-\eta-1}^{t-\eta} x(t')dt',$$

where η is the period of "lag."

This equation may be solved in a series of characteristic terms in the same way as (40) by using the auxiliary difference-differential equation

$$(47) \qquad \dot{y}(t) = h\{y(t-\eta) - y(t-\eta-1)\}.$$

The continuous solution of (47) is determined once the constraint is specified in a range of length $\eta+1$, say $(0, \eta+1)$. By the methods developed earlier we find that such a continuous solution may be defined by the series (18), where

$$(48)\qquad k_n = \frac{\rho_n \int_0^{\eta+1} y(t)e^{-\rho_n t}dt - he^{-\eta\rho_n}\int_0^1 y(t)e^{-\rho_n t}dt + y(\eta+1)e^{-\rho_n(\eta+1)}}{1 + h\eta e^{-\rho_n\eta} - h(\eta+1)e^{-\rho_n(\eta+1)}}$$

and $F_0(t)$ is equal to $A+Be^{\alpha t}$ when $h\neq 1$, and to $A+Bt$ when $h=1$.

If this series is uniformly convergent in the range $(0, \infty)$, it has a derived series uniformly convergent in the range $(\eta+1, \infty)$, so that a solution of (46), $x(t)=\dot{y}(t)$, is given by the equation

$$(49)\qquad x(t) = F_0'(t) + \sum_1^\infty (k_n\rho_n e^{\rho_n t} + \bar{k}_n\bar{\rho}_n e^{\bar{\rho}_n t}),$$

in the range $(\eta+1, \infty)$ of t.

The characteristic solutions of (46) are found by substituting $ke^{\rho t}+\bar{k}e^{\bar{\rho}t}$ for $x(t)$ in (46). We thus find, as the equation for the characteristic numbers,

$$(50)\qquad \rho = h(e^{-\eta\rho} - e^{-(\eta+1)\rho}).$$

There is a characteristic number of (50) in each of the ranges $\left(\frac{2n\pi}{\eta+1}, \frac{2n\pi+2\pi}{\eta+1}\right)$ of u, $(n=1, 2, \cdots)$. The asymptotic expression for ρ_n is

$$\rho_n \approx -\frac{1}{1+\eta}\log_e\left\{\frac{(2n+\frac{1}{2})\pi}{h(1+\eta)}\right\} + \frac{(2n+\frac{1}{2})\pi i}{(1+\eta)}$$

when h is positive, and

$$\rho_n \approx -\frac{1}{1+\eta}\log_e\left\{\frac{(2n+\frac{3}{2})\pi}{|h|(1+\eta)}\right\} + \frac{(2n+\frac{3}{2})\pi i}{(1+\eta)}$$

when h is negative.

9. The first-order difference-differential equation (2) occurs most frequently in economic applications, but sometimes a second-order derivative appears, so that some attention might be directed to the equation

$$(51)\qquad \lambda\ddot{y}(t) - \dot{y}(t) + ay(t) - cy(t-1) = 0,$$

which reduces to (2) when $\lambda=0$.

A solution of (51) is completely determined, (a), when an integrable constraint is specified for the range (0, 1), say, and (b), when two values of $y(t)$ are given in each of the unit ranges $(r, r+1)$, $(r=1, 2, \cdots)$. It is evident, then, that there is not a unique "continuous" solution, for the condition of continuity specifies only one constant of integration whereas there are two arbitrary constants involved in the solution of any second-order differential equation. There is, however, a unique solution which is continuous, together with its first derivative, so that

$y(t+0)=y(t-0)$, $\dot{y}(t+0)=\dot{y}(t-0)$. These two conditions are sufficient, together with the form of the constraint, to determine completely a solution of (51), and in the present Section it is proposed to discuss the formulation of this particular solution as an infinite series of characteristic terms.

Expocyclic characteristic solutions of (51) are investigated by assuming the substitution $y(t)=ke^{\rho t}+\bar{k}e^{\bar{\rho}t}$. It is found that the characteristic numbers satisfy the relation

$$ce^{-\rho} = a - \rho + \lambda\rho^2. \tag{52}$$

There is one characteristic number of (52) in each of the ranges $(2n\pi, 2n\pi+2\pi)$ of u. The asymptotic value of the nth characteristic number is

$$\rho_n \approx -\log_e\left(\frac{\lambda}{c}\right) - 2\log_e(2n+1)\pi + (2n+1)\pi i \tag{53}$$

when (λ/c) is positive, and

$$\rho_n \approx -\log_e\left|\frac{\lambda}{c}\right| - 2\log_e 2n\pi + 2n\pi i \tag{54}$$

when (λ/c) is negative.

In addition to the expocyclic solutions there are nonoscillatory solutions, given by the real solutions of (52). We shall only quote briefly for the case $a=c=1$. The most general linear combination of characteristic solutions is

$$F_0(t) + \sum_1^{\infty}(k_n e^{\rho_n t} + \bar{k}_n e^{\bar{\rho}_n t}), \tag{55}$$

where $F_0(t)=A+Bt+Ce^{\alpha t}$ when $\lambda\neq\frac{1}{2}$, and $A+Bt+Ct^2$ when $\lambda=\frac{1}{2}$.

It may be established by the methods developed in earlier Sections that a solution of the equation (51) can be expanded in a series of the form (55), the expansion being valid for the range $(0, \infty)$. The k's are given by

$$k_n=\frac{(a-\rho_n+\lambda\rho_n^2)\int_0^1 y_0(t)e^{-\rho_n t}dt-(1-\lambda\rho_n)y_0(1)e^{-\rho_n}+\lambda\dot{y}_0(1)e^{-\rho_n}}{(ce^{-\rho_n}-1+2\lambda\rho_n)}, \tag{56}$$

except in the case of a double root, as happens when $a=c=1$, there being in this case a double root at $\rho_0=0$. The method of calculating the residue of a function with a multiple pole, referred to in an earlier Section, here leads to the expression for $F_0(t)$, $A+Bt+Ce^{\alpha t}$, quoted before in the case where $\lambda\neq\frac{1}{2}$. When $\lambda=\frac{1}{2}$, there is a triple root, and we find for $F_0(t)$ an expression of the form $A+Bt+Ct^2$.

The integro-differential equation

$$\lambda\dot{x}(t) - x(t) + h\int_{t-1}^{t} x(t')dt' = 0 \tag{57}$$

is allied to the equation (51) in the same way as equations (40) and (41) are related. The substitution $x(t) = \dot{y}(t)$ in (57) reproduces (51). It is possible to show that if the series (55) is uniformly convergent in the range $(0, \infty)$, its derived series is uniformly convergent in the range $(\frac{1}{2}, \infty)$, so that we may write for a solution of (57),

$$x(t) = F_0'(t) + \sum_{1}^{\infty} (k_n\rho_n e^{\rho_n t} + \bar{k}_n\bar{\rho}_n e^{\bar{\rho}_n t}), \tag{58}$$

the expansion being valid in the range $(\frac{1}{2}, \infty)$ in general. Under special circumstances, of course, the range within which this expansion is valid is more extensive.

We desire to express our sincere thanks to Professor J. H. Michell, F.R.S. and to Professor T. M. Cherry, Ph.D. for helpful criticisms during the course of the present investigation.

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Errata

Attention is drawn to a misprint in our paper, "The Influence of Distributed Lags," published in Econometrica, Vol. 6, p. 159. The equation at the top of page 161 should read:

$$I(t) - U = Ce^{-0.024t} \cos (0.625t + \phi).$$

In the expression for $J(t)$ on p. 162 read also

$$J(t) = Ce^{0.036t} \cos (0.616t + \phi).$$

THE EMPIRICAL IMPLICATIONS OF UTILITY ANALYSIS

By Paul A. Samuelson

I

It is more than half a century since the first formulations of utility analysis by Jevons, Menger, and Walras. In that time there has been much controversy for and against this concept.

Although much of the discussion has not gone beyond a quasi-philosophical defense or rejection of the utility concept, it is nevertheless possible to discern clear lines of development in the literature. First, there has been a steady tendency toward the removal of moral, utilitarian, welfare connotations from the concept. Secondly, there has been a progressive movement toward the rejection of hedonistic, introspective, psychological elements. These tendencies are evidenced by the names suggested to replace utility and satisfaction—ophélimité, desirability, wantability, etc.

The question arises as to what is left when all these elements are removed. Does not the whole utility analysis become *meaningless* in the operational sense of modern science? A meaningless theory according to this criterion is one which has no empirical implications by which it could conceivably be refuted under ideal empirical conditions. Thus, it is meaningless to ask whether the earth *really* moves around the sun rather than the sun around the earth, since no hypothesis with respect to the facts of celestial behavior is implied by either of these conventions. Is the same true of utility analysis? Has it no empirical implications for price-quantity behavior?

It is clear that in its early formulations it was thought to have very definite, even revolutionary, consequences for the analysis of price and value. Moreover, even today the instinct of the textbook writer is methodologically sound in his attempt to deduce the negatively sloping demand curve from the Weber-Fechner law and diminishing marginal utility; this does not alter the fact that the whole demonstration is hopelessly fallacious and illogical.

That some modern formulations of the utility concept are empty, circular, and meaningless in the above sense, is hardly open to doubt.[1] Consider, for example, a typical view as follows. (1) People act according to a plan; (2) a plan is how people act; (3) hence, people act as

[1] For an illuminating survey of the present status of utility theory see Alan R. Sweezy, "The Interpretation of Subjective Value Theory in the Writings of the Austrian Economists," *Review of Economic Studies*, Vol. 1, June, 1934, pp. 176–185.

they act. This last conclusion is devoid of empirical significance, being consistent with any and all kinds of behavior.

It is the purpose here to demonstrate that the utility analysis in its ordinary form does contain empirically meaningful implications by which it could be refuted. Whether the hypothesis with respect to price-quantity behavior implied in the utility analysis is fruitful or not is a question which lies outside the province of this paper. It is the sole purpose of this paper to finish a much-neglected task, that of squeezing out of the utility analysis its empirical implications for individual and group price-quantity behavior.

Only the most general assumptions are made: that there exists an ordinal preference field satisfying everywhere curvature conditions sufficient to insure a proper relative maximum under the constraint of a fixed total budget.

Mathematically, our ordinal preference field may be written as a function of the n commodities $(x_1, \cdots, x_n)$,

$$U = F[\phi(x_1, \cdots, x_n)], \qquad \frac{dF}{d\phi} > 0,$$

where ϕ represents any one utility index and U any other utility index.

For any particular utility index the curvature conditions sufficient for a relative maximum under the budgetary restraint

$$x_1p_1 + x_2p_2 + \cdots + x_np_n = I$$

are known to be[2]

$$\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2\phi}{\partial x_i\partial x_j}\xi_i\xi_j < 0,$$

for

$$\sum_{i=1}^{n} \frac{\partial\phi}{\partial x_i}\xi_i = 0, \qquad \text{not all } \xi\text{'s} = 0.$$

The restriction upon price-quantity behavior from which all our theorems will be derived will be indicated later.

[2] This condition is invariant under any transformation of the utility index, for

$$\frac{\partial^2 U}{\partial x_i\partial x_j} = F'\frac{\partial^2\phi}{\partial x_i\partial x_j} + F''\frac{\partial\phi}{\partial x_i}\frac{\partial\phi}{\partial x_j}.$$

Hence,

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\left(F'\frac{\partial^2\phi}{\partial x_i\partial x_j} + F''\frac{\partial\phi}{\partial x_i}\frac{\partial\phi}{\partial x_j}\right)\xi_i\xi_j = F'\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial^2\phi}{\partial x_i\partial x_j}\xi_i\xi_j + F''\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial\phi}{\partial x_i}\frac{\partial\phi}{\partial x_j}\xi_i\xi_j.$$

But for $\sum_{i=1}^{n}\frac{\partial U}{\partial x_i}\xi_i = F'\sum_{i=1}^{n}\frac{\partial\phi}{\partial x_i}\xi_i = 0$ the second term vanishes, and

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial^2 U}{\partial x_i\partial x_j}\xi_i\xi_j < 0 \qquad \text{for} \qquad \sum_{i=1}^{n}\frac{\partial U}{\partial x_i}\xi_i = 0.$$

II

It has long been recognized that under very general assumptions it is possible to rule out definitely many kinds of consumer's behavior in response to price changes. Barring changes in data (i.e., shifts in demand schedules), it was commonly believed that the quantity of product demanded by an individual or market would necessarily increase with a lowering of its price. For example, we have Marshall's dictum:

> Every fall, however slight in the price of a commodity in general use, will, other things being equal, increase the total sales of it.[3]

This was felt to be implied by the diminishing marginal utility of the good, and conversely.[4]

More recent investigators such as Allen, Georgescu-Roegen, Hicks, Hotelling, Johnson, Schultz, and Slutsky have concerned themselves with the implications for consumer's behavior of the conventional utility- or indifference-curve assumptions.

Recently I proposed a new postulational base upon which to construct a theory of consumer's behavior.[5] It was there shown that from this starting point could be erected a theory which included all the elements of the previous analysis. There I expressed my opinion as to the advantages from a methodological point of view of such a reorientation. Completely without prejudice to such considerations I should like here to indicate the mathematical simplicities which suggested themselves from that investigation. That is to say, even within the framework of the ordinary utility- and indifference-curve assumptions it is believed to be possible to derive already known theorems quickly, and also to suggest new sets of conditions. Furthermore, by means of this approach the transition from individual- to market-demand functions is considerably expedited.

III

The essential advantages referred to above may be secured by writing the "stability" conditions for an individual implied in all utility- and indifference-curve analyses in the following simple form:[6]

$$\sum_{i=1}^{n} \Delta p_i \Delta x_i < 0, \tag{1}$$

[3] A. Marshall, *Principles of Economics*, p. 98.

[4] One exception was recognized, however, and received the title of Giffen's Paradox.

[5] P. A. Samuelson, "New Foundations for the Pure Theory of Consumer's Behaviour," *Economica*, Vol. 5 (New Series), February, 1938, pp. 61–71.

[6] *Ibid.* Also Georgescu-Roegen, "The Pure Theory of Consumer's Behavior," *Quarterly Journal of Economics*, Vol. 50, August, 1936, pp. 545–593.

for

$$\sum_{i=1}^{n} p_i \Delta x_i = 0 \qquad \text{and not all } \Delta x_i = 0,$$

where $(x_1, \cdots, x_n)$ are respective consumer's goods and $(p_1, \cdots, p_n)$ their prices. More generally, we may include productive services sold by the individual as consumer's goods if we agree upon the convention of regarding services sold as the negative of commodities (services) bought.

By going through a limiting process, we may rewrite condition (1) as follows:

$$\sum_{i=1}^{n} dp_i dx_i < 0, \tag{2}$$

for

$$\sum_{i=1}^{n} p_i dx_i = 0 \qquad \text{and not all } dx_i = 0.$$

It will be noted that only prices and quantities, observable phenomena, appear in these expressions.

From the condition that utility be a maximum, or that the individual be at a point of highest preference subject to a fixed total expenditure and given prices, it is possible to derive for any individual the following demand functions:

$$\begin{aligned} x_1 &= h^1(p_1, \cdots, p_n, I), \\ x_2 &= h^2(p_1, \cdots, p_n, I), \\ &\cdot \\ &\cdot \\ &\cdot \\ x_n &= h^n(p_1, \cdots, p_n, I), \end{aligned} \tag{3}$$

where

$$I = \sum_{i=1}^{n} x_i p_i. \tag{4}$$

In fact, the whole object of the utility analysis is the attainment of these functions. It will be recalled from the ordinary rules of differential calculus that

$$dx_i = \sum_{j=1}^{n} h_j^i dp_j + h_I^i dI, \qquad (i = 1, \cdots, n), \tag{5}$$

when

$$h_j^i = \frac{\partial x_i}{\partial p_j}$$

and

$$h_I^i = \frac{\partial x_i}{\partial I}\,.$$

Also

$$dI = \sum_{i=1}^{n} x_i dp_i + \sum_{i=1}^{n} p_i dx_i. \tag{6}$$

Therefore, we may rewrite (5) as

$$dx_i = \sum_{j=1}^{n} h_j{}^i dp_j + h_I{}^i \left(\sum_{j=1}^{n} x_j dp_j + \sum_{j=1}^{n} p_j dx_j \right). \tag{7}$$

But it is part of our hypothesis in (2) that

$$\sum_{j=1}^{n} p_j dx_j = 0.$$

Therefore, substituting in (2), we have

$$\sum_{i=1}^{n} \sum_{j=1}^{n} (h_j{}^i + x_j h_I{}^i) dp_i dp_j < 0, \quad (i = 1, \cdots, n), \tag{8}$$

for not all $dx_i = 0$; or what comes to the same thing, for not all dp_i/p_i equal.

The condition expressed in (8) is a very simple one and has not to date, I believe, appeared in the literature. This quadratic form, symmetrical as a condition of integrability as pointed out by Slutsky, is semi-definite because of the homogeneity condition that doubling all prices and incomes leaves all quantities invariant. However, the quadratic form made up of any subset of $(n-1)$ variables must be negative definite. A necessary and sufficient condition that this be so is that the principal minors of order m of the following determinant have the sign $(-1)^m$. Let

$$\Delta = \left| \, h_j{}^i + x_j h_I{}^i \, \right|,$$

$$\Delta_{ij} = \begin{vmatrix} h_i{}^i + x_i h_I{}^i & h_j{}^i + x_j h_I{}^i \\ h_j{}^i + x_j h_I{}^i & h_j{}^j + x_j h_I{}^j \end{vmatrix}; \text{ etc.}$$

Then

$$(-1)^m \Delta_{i_1}, \cdots, {}_{i_m} > 0, \qquad m \leqq n - 1. \tag{9}$$

More specifically, we must have the following conditions:

$$\frac{\partial x_i}{\partial p_i} + x_i \frac{\partial x_i}{\partial I} < 0, \tag{10}$$

$$\left(\frac{\partial x_i}{\partial p_i} + x_i \frac{\partial x_i}{\partial I}\right)\left(\frac{\partial x_j}{\partial p_j} + x_j \frac{\partial x_j}{\partial I}\right) - \left(\frac{\partial x_i}{\partial p_j} + x_j \frac{\partial x_i}{\partial I}\right)^2 > 0, \quad \text{etc.}[7] \tag{11}$$

[7] Samuelson, *loc. cit.*, p. 69.

It is not possible then to state that the elasticity of a good with respect to its own price must have an algebraically negative sign. This need not be so, provided that the income elasticity of demand be sufficiently negative, so that the whole term in (10) be less than zero.

These restrictions as developed thus far are seen to involve not only price elasticities of demand, but also the income elasticity of demand. However, from the condition that the demand functions in (3) be homogeneous of order zero, it will generally be true that the income elasticity of demand can be expressed in terms of price elasticities of demand, since a lowering of all prices is equivalent to an increase in money income. Indeed, from Euler's theorem on homogeneous functions:

$$h_1{}^i p_1 + h_2{}^i p_2 + \cdots + h_n{}^i p_n = -h_I{}^i I, \qquad (i = 1, \cdots, n). \tag{12}$$

It should be possible, therefore, to develop restrictions on the demand functions relating only to price elasticities of demand. This can be done without making use of (12) directly as will be shown in the following discussion.

IV

Still considering only a single individual, let us regard his total income, I, as a constant and permit only prices to vary. Therefore,

$$dI = \sum_{i=1}^{n} x_i dp_i + \sum_{i=1}^{n} p_i dx_i = 0. \tag{13}$$

Recalling from hypothesis (2) that

$$\sum_{i=1}^{n} p_i dx_i = 0, \tag{14}$$

we have

$$\sum_{i=1}^{n} x_i dp_i = 0 \tag{15}$$

and

$$\sum_{i=1}^{n} dp_i dx_i < 0, \tag{16}$$

not all $dx_i = 0$.

Setting I at its constant value $\bar{I}$, we may rewrite our demand functions of (3) as follows:

$$x_i = h^i(p_1, \cdots, p_n, \bar{I}), \qquad (i = 1, \cdots, n), \tag{17}$$

where only $(p_1, \cdots, p_n)$ are regarded as variables.
Then

$$dx_i = \sum_{j=1}^{n} h_j{}^i dp_j, \qquad (i = 1, \cdots, n), \tag{18}$$

and (16) becomes

(19)
$$\sum_{i=1}^{n} \sum_{j=1}^{n} h_j^i dp_i dp_j < 0,$$
for
$$\sum_{i=1}^{n} x_i dp_i = 0,$$
not all $dp_i = 0$.

In general,

(20)
$$h_j^i \neq h_i^j,$$

and so it is necessary to define a new symmetrical set of coefficients for this quadratic form:

(21)
$$\beta_{ij} = \frac{h_j^i + h_i^j}{2} = \beta_{ji}.$$

Thus we may rewrite (19) as

(22)
$$\sum_{i=1}^{n} \sum_{j=1}^{n} \beta_{ij} dp_i dp_j < 0,$$
for
$$\sum_{i=1}^{n} x_i dp_i = 0,$$
not all $dp_i = 0$.

Consider the following determinant:

(23)
$$D = \left| \begin{array}{c|c} \beta_{ij} & \begin{matrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{matrix} \\ \hline x_1 \cdots x_n & 0 \end{array} \right|.$$

Let

(24)
$$D_{12} = \begin{vmatrix} \beta_{11} & \beta_{12} & x_1 \\ \beta_{21} & \beta_{22} & x_2 \\ x_1 & x_2 & 0 \end{vmatrix}, \qquad D_{123} = \begin{vmatrix} \beta_{11} & \beta_{12} & \beta_{13} & x_1 \\ \beta_{21} & \beta_{22} & \beta_{23} & x_2 \\ \beta_{31} & \beta_{32} & \beta_{33} & x_3 \\ x_1 & x_2 & x_3 & 0 \end{vmatrix},$$

and likewise for other combinations of subscripts.

Our conditions of (22) and (19) are that any one of the determinants $D_{i_1, \ldots, i_{m-1}}$ must have the same sign as $(-1)^{m-1}$ where m is the order of the determinant involved; i.e.,

(25)
$$(-1)^{m-1} D_{i_1, \ldots, i_{m-1}} > 0, \qquad m \leqq n + 1.$$

Specifically,

$$h_i^i(x_j)^2 - (h_j^i + h_i^j)x_jx_i + h_j^j(x_i)^2 < 0 \tag{26}$$

for any $i \neq j$, etc.

Professor Hotelling has developed similar conditions for the inverse of these demand functions.[8]

Consider the set of demand functions in (17):

$$x_i = h^i(p_1, \cdots, p_n, I), \qquad (i = 1, \cdots, n).$$

These are n equations in $2n$ unknowns. In general, it is possible to solve these inversely for the p's in terms of the x's as follows:

$$p_i = g^i(x_1, \cdots, x_n, I), \qquad (i = 1, \cdots, n). \tag{27}$$

Therefore,

$$dp_i = \sum_{j=1}^{n} g_j^i dx_j, \qquad (i = 1, \cdots, n). \tag{28}$$

We may rewrite, therefore, (16) as follows:

$$\sum_{i=1}^{n} \sum_{j=1}^{n} g_j^i dx_i dx_j < 0, \tag{29}$$

for

$$\sum_{i=1}^{n} p_i dx_i = 0.$$

Defining

$$\gamma_{ij} = \frac{g_j^i + g_i^j}{2} = \gamma_{ji}, \tag{30}$$

we may rewrite (29) as

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \gamma_{ij} dx_i dx_j < 0, \qquad \text{for} \qquad \sum_{i=1}^{n} p_i dx_i = 0, \tag{31}$$

not all $dx_i = 0$.

Consider the determinants

$$E = \begin{vmatrix} & & & p_1 \\ & \gamma_{ij} & & \vdots \\ & & & p_n \\ p_1 & \cdots & p_n & 0 \end{vmatrix} \tag{32}$$

and

$$E_{ij} = \begin{vmatrix} \gamma_{ii} & \gamma_{ij} & p_i \\ \gamma_{ji} & \gamma_{jj} & p_j \\ p_i & p_j & 0 \end{vmatrix}, \text{ etc.} \tag{33}$$

[8] H. Hotelling, "Demand Functions with Limited Budgets," ECONOMETRICA, Vol. 3, January, 1935, pp. 66–78.

Conditions (29) and (31) require that any of the determinants $E_{i_1}, \ldots, {}_{i_{m-1}}$ must have the same sign as $(-1)^{m-1}$ where m is the order of the determinant involved; i.e.,[9]

$$(34) \qquad (-1)^{m-1}E_{i_1}, \ldots, {}_{i_{m-1}} > 0, \qquad m \leqq n+1.$$

In one important case it is impossible to solve equations in (17) inversely for (27). This is in the case where total expenditure is equal to zero. It was this case where the total value of services rendered or of goods given up by an individual was just balanced by total value of goods consumed which received great attention from the older mathematical economists. Here (17) becomes

$$(35) \qquad x_i = h^i(p_1, \cdots, p_n, 0), \qquad (i = 1, \cdots, n).$$

Since these expressions are homogeneous of order zero, even when I is held constant at the level zero, the Jacobian

$$(36) \qquad J = |h_j^i| = 0, \qquad \text{for } I = 0,$$

and the inverse solution is not possible. This is clearly seen in Hotelling's presentation, where for $I=0$, the functions in (27) become indeterminate. However, the conditions given here in (25) are still valid for this important case.

V

It remains now to consider relations between individual demand functions and the general demand functions for the whole market. Let us suppose we have s individuals in the market place and that the demand functions of the rth individual may be written

$$(37) \qquad {}^rx_i = {}^rh^i(p_1, \cdots, p_n, {}^rI), \qquad (i = 1, \cdots, n).$$

The total amount demanded of a good is equal to the sum of all individual amounts:

$$(38) \qquad X_i = {}^1x_i + {}^2x_i + \cdots + {}^sx_i = \sum_{r=1}^{s} {}^rx_i, \quad (i=1, \cdots, n).$$

Thus we may write the general demand functions as follows:

$$(39) \qquad X_i = H^i(p_1, \cdots, p_n, {}^1I, \cdots, {}^sI), \qquad (i = 1, \cdots, n),$$

$$(40) \qquad = \sum_{r=1}^{s} {}^rh^i(p_1, \cdots, p_n, {}^rI).$$

It will be noted that the general amount of a commodity demanded is a function of all prices and the incomes of each and every person in the market place.

[9] *Ibid.*, p. 72.

Obviously,

$$\frac{\partial X_i}{\partial p_j} = H_j^i = \sum_{r=1}^{s} {}^rh_j^i. \tag{41}$$

Let us consider first the conditions of Section IV, where all incomes are constant, i.e.,

$$^rI = \overline{^rI}, \qquad (r = 1, \cdots, s). \tag{42}$$

Then

$$^rx_i = {}^rh^i(p_1, \cdots, p_n, \overline{^rI}), \qquad (i = 1, \cdots, n), \tag{43}$$

and

$$X_i = H^i(p_1, \cdots, p_n, \overline{^1I}, \cdots \overline{^sI}), \tag{44}$$

$$= \sum_{r=1}^{s} {}^rh^i(p_1, \cdots, p_n, \overline{^rI}). \tag{45}$$

For each individual,

$$d(^rI) = 0 = \sum_{i=1}^{n} {}^rx_i dp_i + \sum_{i=1}^{n} p_i d(^rx_i). \tag{46}$$

And for

$$\sum_{i=1}^{n} p_i d(^rx_i) = 0 \tag{47}$$

we have

$$\sum_{i=1}^{n} {}^rx_i dp_i = 0 \tag{48}$$

and

$$\sum_{i=1}^{n} d(^rx_i) dp_i < 0. \tag{49}$$

Since this holds for each, we may sum over all to get

$$\sum_{i=1}^{n} dp_i \sum_{r=1}^{s} d(^rx_i) = \sum_{i=1}^{n} dp_i dX_i < 0, \tag{50}$$

for

$$\sum_{i=1}^{n} X_i dp_i = 0 \qquad \text{or} \qquad \sum_{i=1}^{n} p_i dX_i = 0.$$

But, from (44),

$$dX_i = \sum_{j=1}^{n} H_j^i dp_j, \tag{51}$$

and so (50) becomes

$$\sum_{i=1}^{n}\sum_{j=1}^{n} H_j^i dp_i dp_j < 0 \tag{52}$$

for

$$\sum_{i=1}^{n} p_i dX_i = 0,$$

not all $dX_i = 0$,
or

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{H_j^i + H_i^j}{2}\right) dp_i dp_j < 0, \tag{53}$$

for

$$\sum_{i=1}^{n} p_i dX_i = 0,$$

not all $dX_i = 0$.

Consider the determinants

$$F = \begin{vmatrix} \frac{H_j^i + H_i^j}{2} & \begin{matrix} X_1 \\ \cdot \\ \cdot \\ X_n \end{matrix} \\ X_1 \cdots X_n & 0 \end{vmatrix} \tag{54}$$

and

$$F_{ij} = \begin{vmatrix} H_i^i & \frac{H_j^i + H_i^j}{2} & X_i \\ \frac{H_j^i + H_i^j}{2} & H_j^j & X_j \\ X_i & X_j & 0 \end{vmatrix}, \text{ etc.} \tag{55}$$

Then every such determinant $F_{i_1}, \cdots, {}_{i_{m-1}}$ must have the same sign as $(-1)^{m-1}$; or

$$(-1)^{m-1}F_{i_1,\cdots,i_{m-1}} > 0, \qquad m \leqq n+1. \tag{56}$$

Specifically,

$$H_i^i(X_j)^2 + H_j^j(X_i)^2 - (H_j^i + H_i^j)X_iX_j < 0, \text{ etc.} \tag{57}$$

Of course this does *not* imply that partial-equilibrium demand curves must be negatively inclined. Furthermore, the analysis holds for all total (net) incomes equal to zero.

Professor Hotelling in his analysis of this same problem has indicated his belief that more rigid conditions can "with considerable probability be supposed to hold for total demand functions."[10] This may or may not be so, but it is not the part of mathematical deductive analysis to answer this question. Suffice it to point out contrary examples, such as negatively inclined supply schedules, etc.

One more point remains to be discussed, that of extending the conditions of Section III from individual to general demand schedules. Recall that for each individual

$$\sum_{i=1}^{n}\sum_{j=1}^{n}({}^{r}h_j{}^{i} + {}^{r}x_j{}^{r}h_I{}^{i})dp_i dp_j < 0, \qquad (r = 1, \cdots, s), \tag{58}$$

for not all prices changed proportionately. Summing with respect to all individuals, we get

$$\sum_{i=1}^{n}\sum_{j=1}^{n}(H_j{}^{i} + \sum_{r=1}^{s} {}^{r}x_j{}^{r}h_I{}^{i})dp_i dp_j < 0, \tag{59}$$

for not all prices changing proportionately. As before the integrability conditions require that these coefficients be symmetrical with respect to i and j.

Call

$$\begin{aligned} \alpha_{ij} &= H_j{}^{i} + \sum_{r=1}^{s}({}^{r}x_j{}^{r}h_I{}^{i}) = \alpha_{ji}, \\ B &= \left| \alpha_{ij} \right|, \\ B_{ij} &= \begin{vmatrix} \alpha_{ii} & \alpha_{ij} \\ \alpha_{ji} & \alpha_{jj} \end{vmatrix}; \text{ etc.} \end{aligned} \tag{60}$$

Our final conditions are that each sub-determinant $B_{i_1, \ldots, i_m}$ have the sign $(-1)^m$, or

$$(-1)^m B_{i_1, \ldots, i_m} > 0, \qquad m \leqq n - 1. \tag{61}$$

It will be readily recognized that there is no difficulty in deriving integrability conditions on the general demand functions. The difficulty is in deriving conditions which depend only upon price changes and changes in *total* income. That this is impossible follows from the impossibility, in general, of finding any functional relationship between amounts demanded and *total* income (as distinct from the specification of each and every individual's income). The problems which this entails must be reserved for consideration at a future time.

[10] *Ibid.*, p. 76.

VI

The value of any technique of analysis must depend pragmatically upon the fruits it yields. The easy derivation of conditions (9), (25), (34), (56), and (61) is submitted as evidence of the mathematical advantages of the approach here suggested.

In conclusion, a few remarks may be in order concerning the history of these conditions. Disregarding special unnecessarily restrictive assumptions made by particular authors (Marshall, Pareto, *et al.*), it is surprising how little fruit in the way of demand conditions the analysis of consumer's behavior has afforded. The conditions in (9) were developed for two commodities by W. E. Johnson in 1913.[11] (It may be of significance that these were first discovered by the indifference-curve approach.) Condition (10), one fragmentary part of the conditions in (9), was also pointed out by Slutsky in 1915.[12] Condition (34) was developed by Hotelling in 1935.[13] I am not aware that any of the other conditions have previously appeared in the literature.

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[11] W. E. Johnson, "The Pure Theory of Utility Curves," *Economic Journal*, Vol. 23, December, 1913, pp. 483–513.

[12] E. Slutsky, "Sulla Teoria del Bilancio del Consumatore," *Giornale degli Economisti*, Vol. 51, No. 1, pp. 1–26.

[13] Hotelling, *loc. cit.*

MARSHALL'S PARADOX AND THE DIRECTION OF SHIFT IN DEMAND

By Acheson J. Duncan

Some years ago Alfred Marshall laid down a proposition in the theory of international trade that appears to be paradoxical. He stated that if trade is carried on in two commodities between two countries, say England and Germany, and if England's import-demand for German goods increases, the more elastic England's demand, the elasticity of Germany's demand being given, "the larger will be the volumes both of her (England's) exports and of her imports; but the more also will her exports be enlarged relatively to her imports; or, in other words, the *less favourable* to her will be the terms of trade."[1] This means that the more elastic England's demand the *greater* will be the adverse change in the price of her imports. In ordinary demand analysis, the more elastic the demand, the *less* is the change in price that is associated with a given change in quantity. Marshall's proposition would therefore appear to be contrary to general expectations. Because of this it has recently caught the attention of students of international trade and a controversy has arisen concerning its validity.

The controversy over Marshall's paradox, however, has a much broader aspect than that of merely determining the validity of an abstract proposition in the theory of international trade. The majority of the participants in the argument employed graphic methods, and a review of the various analyses shows that the difference in results obtained is fundamentally related to the way in which "an increase in demand" is graphically represented. Essentially the controversy offers an excellent illustration of the need of distinguishing the direction of shift in demand in the graphic analysis of certain theoretical problems —a need that has already been suggested by certain contributors to this journal.[2]

It is the major purpose of this paper to investigate this larger question. After reviewing the controversy over Marshall's proposition to show the need of distinguishing shifts in demand in different directions, it will venture into the broader problem concerning the nature of the factors determining the direction of shift in demand.

I. THE CONTROVERSY OVER MARSHALL'S PARADOX

A review of the controversy over Marshall's paradox may well begin

[1] Alfred Marshall, *Money, Credit, and Commerce*, p. 178 (italics mine).

[2] See H. P. Hartkemeier, "Note on Shifts in Demand and Supply Curves," Econometrica, Vol. 3, pp. 428–434, and G. Shepherd, "Vertical and Horizontal Shifts in Demand Curves," Econometrica, Vol. 4, pp. 361–367.

with Marshall's own argument. The method of analysis which he employed was a graphic method which is today associated with his name. The type of graphs used is pictured in Figure 1. Here the curve OG represents Germany's import-demand (her export-supply), and the curve OE represents England's import-demand (her export-supply) in its initial state. The point of intersection P is the initial point of equilibrium. At this point England's imports are measured by ON, her exports by OM, and the terms of trade by OM/ON, i.e., by the tangent of the angle PON.[3] Germany's imports are measured by OM and her exports by ON.

Marshall indicates an increase in England's demand for German goods by shifting the English demand curve to the right. For a curve of given elasticity, the new state of demand is represented by the curve OE'. A curve of greater elasticity would be less steep at the initial point of equilibrium and, when shifted to the right, would take the position indicated by OE''. The points P'' and P' are the new points of equilibrium for the curves of greater and less elasticity respectively; OM'' and ON'' and OM' and ON' represent the quantities of each good traded at these points. Marshall's proposition follows from the fact that OM'' and ON'' are greater than OM' and ON' respectively and that the difference $OM''-OM'$ is relatively greater than the difference $ON''-ON'$, indicating that the terms of trade would be *less* favorable to England if her demand were more elastic.[4]

Because this proposition of Marshall's is so contrary to general expectations, Professor F. D. Graham has been led to brand it entirely false. To him Marshall's own definition of elasticity implied that as the terms of trade went against England because of the increase in her demand for German goods, England would take a quantity of imports which would vary *inversely* with the elasticity of her demand. He therefore concluded that the correct solution of the problem should be just the opposite of that given by Marshall.[5]

This attack on Marshall's proposition aroused Professor Bresciani-Turroni to its defense. In reply to Graham, he pointed out that the results of Marshall's graphic analysis depended on the fact that Marshall represented the increase in England's demand by an increase in the quantity of exports which would be given for a fixed quantity of

[3] The terms of trade so defined represent the price of German goods in terms of English goods. They might equally well have been defined as ON/OM, in which case they would represent the price of English goods in terms of German goods.

[4] Marshall, *op. cit.*, p. 343.

[5] F. D. Graham, "The Theory of International Values," *Quarterly Journal of Economics*, Vol. 46, pp. 601–602.

imports, i.e., by an increase in the price (terms of trade) which would be paid for those imports. If he had represented the increase in demand

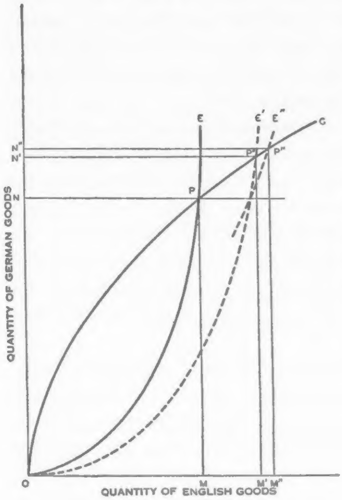

FIGURE 1

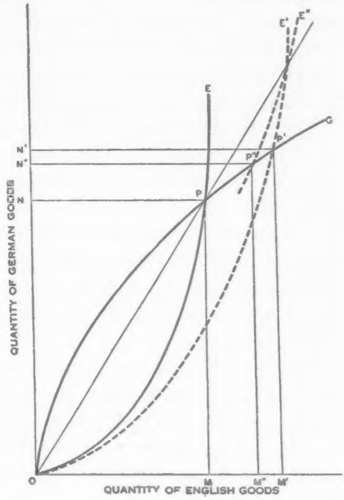

FIGURE 2

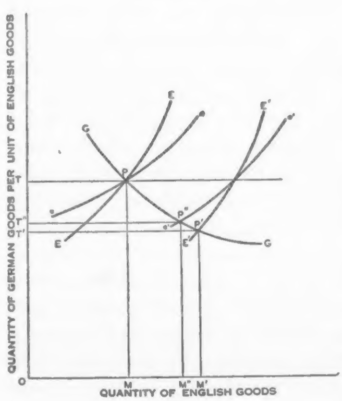

FIGURE 3

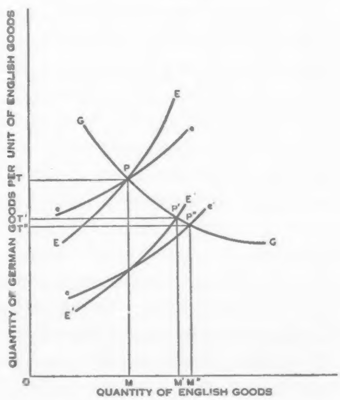

FIGURE 4

by an increase in the quantity of imports which would be taken at a given price (terms of trade), Marshall's conclusions would have been the same as Graham's. This is shown by Figure 2. Here the new de-

mand curves are projected along the price radii instead of along lines parallel to the horizontal axis. In this case, it is seen that the more elastic England's demand, the smaller are the volumes both of her imports and of her exports and the more favorable to her are the terms of trade. It was therefore Bresciani-Turroni's conclusion that both Marshall and Graham were right, the results depending on how the increase in England's demand was graphically interpreted.[6]

Recently Professor Jacob Viner has renewed the attack on Marshall's proposition. Applying a *modified* version of Marshall's graphic analysis, he finds that when the increase in England's demand is represented by a shift in the English curve horizontally to the right, i.e., in the *same direction* in which Marshall shifted his curve, the more elastic England's demand, the smaller are the volumes of her imports and exports and the more favorable to her are the terms of trade. Viner therefore concludes that Graham alone was right and Marshall was wrong.[7]

Since Viner's graphs, however, are not exactly the same as Marshall's, the question arises whether the shift of Viner's curve to the right is the equivalent of a shift of Marshall's curve to the right. The type of graph used by Viner is pictured in Figure 3. It will be noted that the quantity of English goods offered for export is measured horizontally as in Marshall's graphs, but the price (terms of trade) is measured vertically. The quantity of imports which would be taken at any price is represented by the area of the rectangle of which one side is the quantity of goods offered for export and the other side the given price. Viner claims to follow Marshall by shifting the English curve to the right. When this is done, Graham's and not Marshall's solution is found to be correct (see Figure 3). By representing the increase in England's demand, however, as an increase in the quantity of imports which would be taken at a given price, Viner is not using the same method of representing an increase in demand even though the direction of shift is the same. To have used the same method as Marshall, Viner should have represented an increase in demand as an increase in the quantity of exports (i.e., an increase in the price) which would be paid for a given quantity of imports. Owing to the reciprocal nature of demand and supply, an increase in demand for imports is an increase in the supply of exports. To have followed Marshall, therefore, Viner should have represented the increase in England's demand as a decrease in the price at which a given quantity of exports would be offered, i.e., he should have shifted his English curve vertically downwards. If this is done, it is found that the more elastic the English demand (supply)

[6] C. Bresciani-Turroni, "The 'Purchasing Power Parity' Doctrine," *L'Egypte Contemporaine*, Vol. 25, p. 455.

[7] J. Viner, *Studies in the Theory of International Trade*, pp. 543–546.

curve, the greater are the volumes both of England's imports and of her exports, and the less favorable to her are the terms of trade (see Figure 4). In other words, when Marshall's method of representing an increase in demand is followed, Viner's type of graphic analysis leads to the same conclusion as Marshall's.

The controversy over Marshall's proposition thus shows that for a given type of graph the solution depends on the way in which England's demand curve is shifted. It also shows that when two graphs are of different types, a shift in a given direction on one graph is not necessarily the equivalent of a shift in the given direction on the other.

II. AN ALGEBRAIC SOLUTION OF MARSHALL'S PROBLEM

If certain assumptions are made with respect to underlying conditions and their variation, Marshall's problem can be solved algebraically. In the present instance the following assumptions will be made: First it will be assumed that there are only two countries trading in two commodities. Secondly, it will be assumed that the reactions of the individual members of each country to the possession of more or less of the given commodities can be represented collectively by a "national utility function"—a function that will be taken as similar in character to an individual utility function. This assumption may be unrealistic in that it assumes that the collective reactions of a group of individuals will be the same as those of a single average individual. Its use is necessary, however, if the analysis is to be kept relatively simple. The national "utility" function of each country will be assumed to be such that the national preference for more or less of either commodity is not affected by the amount of the other commodity available for consumption, in other words that the commodities are independent.

Thirdly, it will be assumed that the quantity of either commodity which a country must forgo in order to produce a unit of the other commodity remains the same whatever the scale of production of either commodity. This is an assumption of constant costs.

Fourthly, it will be assumed that in the initial state of equilibrium trade is carried on at a ratio of exchange which lies between the ratios of production costs in the two countries. This excludes the special, but nevertheless important, case in which the ratio of exchange coincides with one country's ratio of production costs. The assumption is made, however, in order to insure continuity of the variables in the immediate neighborhood of the initial point of equilibrium.

Finally, it will be assumed that there are no transportation costs between the two countries, that there are no capital movements, and that trade is carried on under conditions of pure competition.

Instead of referring to England and Germany the following abstract symbols will be employed: Let U_t represent the "utility" function of

the tth country, $U_{t.r}$ its first partial derivative with respect to the quantity of the rth commodity consumed by that country, and $U_{t.rr}$ its second partial derivative. Let X_{tr} represent the quantity of the rth commodity consumed in the tth country, X'_{tr} its production in that country, and x_{tr} the quantity of the rth commodity exported therefrom, if positive, or imported thereto, if negative. Finally, let p_r represent the price of the rth commodity in terms of commodity 1. It will help the reader to note that the appearance of a decimal point between subscript numbers indicates a derivative.

The analysis may begin with a description of the conditions characterizing the initial state of equilibrium. These are as follows: First, each country must not find it more desirable, as indicated by its "utility" function, to change the combination in which it is consuming the two commodities through increasing or decreasing its purchases and sales at the equilibrium price. Specifically, the "utility" of the combination being consumed in the stage of equilibrium must be a maximum with respect to the possibility of changing that combination through additional trading at the equilibrium price. The differential "utility" at the equilibrium point must therefore be zero, subject to the condition that $dX_1/dX_2 = p_2$. This characteristic of equilibrium may be described algebraically by the equations $U_{1.2} = p_2U_{1.1}$ and $U_{2.2} = p_2U_{2.1}$. Other characteristics of equilibrium will be an equation between the quantity of each commodity consumed by each country and the quantity produced plus (algebraically) the quantity exported or imported; an equation between the value of its imports and exports; and an equation between the quantity of each commodity exported from one country and the quantity imported into the other.

These characteristics of equilibrium may be summarized by the following set of equations:

$$(1) \quad \begin{cases} U_{1.2} = p_2U_{1.1}, \\ X'_{11} = X_{11} + x_{11}, \\ X'_{12} = X_{12} + x_{12}, \\ x_{11} + p_2x_{12} = 0, \end{cases}$$

$$(2) \quad \begin{cases} U_{2.2} = p_2U_{2.1}, \\ X'_{21} = X_{21} + x_{21}, \\ X'_{22} = X_{22} + x_{22}, \\ x_{21} + p_2x_{22} = 0, \end{cases}$$

$$(3) \quad \begin{cases} x_{11} + x_{21} = 0, \\ x_{12} + x_{22} = 0. \end{cases}$$

Equations (1) are the conditions determining the import-demand and export-supply of country 1, and equations (2) are the conditions determining the import-demand and export-supply of country 2. Equations (3) are the equilibrium conditions for an equality between the quantity demanded by one country and the quantity offered by the other. It will be noted that one of these equations may be obtained from an algebraic combination of the others so that only nine of them are independent. It is also to be noted that since we are assuming constant costs and an equilibrium price ratio lying between the national cost ratios, each country will produce only one of the two commodities. To make the analysis more specific, let it be assumed that country 1 produces only commodity 1 and country 2 only commodity 2. Hence in the above equations, X'_{12} and X'_{21} will be zero.

The problem calls for an increase in one country's import-demand. Let this be country 1. Under the simple conditions here assumed, variation in the import-demand of country 1 may result from a change in the tastes of its people or in its productive power. Consider first a change in its demand resulting from an increase in its productive power.

Let the productive power of country 1 undergo a small increase. With reference to the foregoing equations, this may be indicated by an increase in its production of X_1 from X'_{11} to $X'_{11}+dX'_{11}$. As a consequence of this disturbance, the old set of equilibrium conditions will be replaced by a new set. Since it is assumed that the change in productive power is small, the difference between the old set of equilibrium conditions and the new will be given approximately by the following differential equations:[8]

$$
(4) \qquad \begin{aligned}
U_{1.22}d\overline{X}_{12} &= \bar{p}_2 U_{1.11}d\overline{X}_{11} + U_{1.1}d\bar{p}_2,\\
dX'_{11} &= d\overline{X}_{11} + d\bar{x}_{11},\\
0 &= d\overline{X}_{12} + d\bar{x}_{12},\\
d\bar{x}_{11} + \bar{p}_2 d\bar{x}_{12} + \bar{x}_{12}d\bar{p}_2 &= 0,\\
U_{2.22}d\overline{X}_{22} &= \bar{p}_2 U_{2.11}d\overline{X}_{21} + U_{2.1}d\bar{p}_2,\\
0 &= d\overline{X}_{21} + d\bar{x}_{21},\\
0 &= d\overline{X}_{22} + d\bar{x}_{22},\\
d\bar{x}_{21} + \bar{p}_2 d\bar{x}_{22} + \bar{x}_{22}d\bar{p}_2 &= 0,\\
d\bar{x}_{11} + d\bar{x}_{21} &= 0,\\
d\bar{x}_{12} + d\bar{x}_{22} &= 0.
\end{aligned}
$$

[8] This amounts to assuming that for small changes the "utility" functions can be taken as linear.

The bars over the variables in these equations indicate that they pertain to equilibrium values. It will also be noted that $U_{1.12}$ and $U_{2.12}$ do not appear in these differential equations; for, owing to the assumption that X_1 and X_2 are independent commodities in each country, these partial derivatives are zero.

Our problem is particularly concerned with the variations in the imports, exports, and terms of trade of country 1 resulting from its increase in demand, i.e., with $d\bar{x}_{12}$, $d\bar{x}_{11}$, and $d\bar{p}_2$. From equations (4), these are found to have the following values:

$$
\begin{aligned}
d\bar{x}_{12} &= \bar{p}_2 U_{1.11} dX_{11}' \frac{A_{11}}{D}, \\
(5) \qquad d\bar{x}_{11} &= \bar{p}_2 U_{1.11} dX_{11}' \frac{A_{12}}{D}, \\
d\bar{p}_2 &= \bar{p}_2 U_{1.11} dX_{11}' \frac{A_{13}}{D},
\end{aligned}
$$

where

$$
(6) \qquad D = \begin{vmatrix} -U_{1.22} & \bar{p}_2 U_{1.11} & -U_{1.1} \\ U_{2.22} & -\bar{p}_2 U_{2.11} & -U_{2.1} \\ \bar{p}_2 & 1 & \bar{x}_{12} \end{vmatrix}
$$

and A_{11}, A_{12}, and A_{13} are the cofactors of the elements of its first row. It is our task to find how these variations are related to the elasticity of the import-demand of country 1 at the initial point of equilibrium.

The elasticity of demand at any point is dependent on the slope of the curve at that point. As noted above, equations (1) are the determinants of the import-demand of country 1. If these are differentiated with respect to p_2, it will be found that the slope of its import-demand at the initial point of equilibrium is[9]

$$
(7) \qquad \frac{dx_{12}}{dp_2} = \frac{\begin{vmatrix} U_{1.1} & \bar{p}_2 U_{1.11} \\ -\bar{x}_{12} & 1 \end{vmatrix}}{\begin{vmatrix} -U_{1.22} & \bar{p}_2 U_{1.11} \\ \bar{p}_2 & 1 \end{vmatrix}} = \frac{A_{21}}{A_{23}}.
$$

This shows how the slope of the import-demand of country 1, and hence its elasticity, is related to its "utility" function U_1. Our problem may therefore be solved by noting how those variations in the charac-

[9] It is assumed that country 1 always produces at full capacity so that dX_{11}'/dp_2 is zero.

ter of this function which make that demand more elastic also affect the values of $d\bar{x}_{12}$, $d\bar{x}_{11}$, and $d\bar{p}_2$.

In attempting this solution it is well to restrict the discussion to an analysis of normal situations involving only elastic demand curves, or what amounts to the same thing, only positively sloped supply curves.[10] The criteria for normality will thus be $dx_{11}/dp_1>0$ and $dx_{22}/dp_2>0$, where p_1 is equal to $1/p_2$. By differentiation of equations (1) and (2), it may be shown that

$$dx_{11}/dp_1 = \frac{-p_2^2A_{22}}{A_{23}} \text{ and } dx_{22}/dp_2 = \frac{-A_{11}}{A_{13}}$$

at the point of initial equilibrium. Since x_{12} is negative by definition and p_2, $U_{1.2}$, and $U_{2.1}$ are positive, and since by analogy with individual utility functions $U_{1.11}$, $U_{1.22}$, $U_{2.11}$, and $U_{2.22}$ may be taken as negative, it follows that A_{23} and A_{13} are negative. Hence, the criteria for normality at the point of initial equilibrium require that

$$A_{11} > 0 \text{ and } A_{22} > 0. \tag{8}$$

For normal cases as thus defined, the problem has the following solutions. Equation (7) shows that dx_{12}/dp_2 will be greater and consequently the import-demand of country 1 will be *more* elastic[11] at the point of initial equilibrium, if $U_{1.1}$ is larger or if $|U_{1.22}|$ is smaller. It can also be shown from this equation that for normal situations the demand of country 1 will be more elastic if $|U_{1.11}|$ is smaller.[12] Since D is positive for normal situations,[13] equations (5) and (6) show that the larger the value of $U_{1.1}$, the larger will be the value of D and hence the smaller the values of $|d\bar{x}_{12}|$, $d\bar{x}_{11}$, and $d\bar{p}_2$. Similarly, the smaller the absolute value of $U_{1.22}$ the smaller will be the value of D and hence the larger the values of $|d\bar{x}_{12}|$, $d\bar{x}_{11}$, and $d\bar{p}_2$. If both the numerators and de-

[10] Differentiation of the last of equations (1) shows that $dx_{11}/dp_1=p_2^2x_{12}[1+e_1]$ where e_1 is the elasticity of import-demand of country 1 and is equal to $p_2dx_{12}/x_{12}dp_2$. Hence, if $e_1<-1$, then, since x_{12} is taken to be negative, the slope of the export-supply curve of country 1 [i.e., dx_{11}/dp_1] will be positive. Similarly, by differentiating the last of equations (2) it can be shown that if the import-demand of country 2 is elastic, then its export-supply curve will be positively sloped. Here p_1 is taken equal to $1/p_2$.

[11] Note that dx_{12}/dp_2 is positive since x_{12} is negative, and that the elasticity at the equilibrium point varies directly with dx_{12}/dp_2.

[12] Suppose that $|U_{1.11}|$ decreases by an amount k. Then dx_{12}/dp_2 will *increase* if $dx_{12}/dp_2>-kx_{12}p_2/kp_2^2$, or if $p_2dx_{12}/x_{12}dp_2<-1$, i.e., if the import-demand is elastic.

[13] $D=-U_{1.22}A_{11}+\bar{p}_2U_{1.11}A_{12}-U_{1.1}A_{13}$. Since A_{11} is positive for normal situations and since $U_{1.22}$, $U_{1.11}$, A_{12}, and A_{13} are negative and $U_{1.1}$ is positive, D is positive.

nominators of the expressions for $d\bar{x}_{12}$, $d\bar{x}_{11}$, and $d\bar{p}_2$ [see equations (5)] are divided by $U_{1.11}$, the numerators are freed of this factor and the denominators become $D/U_{1.11}$. But $D/U_{1.11}=\bar{p}_2A_{12}+[-\bar{p}_2U_{2.11}A_{22}+A_{32}]/U_{1.11}$. Hence, since A_{22} is positive for normal situations and since A_{32} is positive, it follows that the smaller the absolute value of $U_{1.11}$ the larger will be the absolute value of $D/U_{1.11}$ and hence the smaller the values of $|d\bar{x}_{12}|$, $d\bar{x}_{11}$, and $d\bar{p}_2$.

These results may be summarized as follows. If the increase in the import-demand of country 1 results from an enlargement of its productive power, the more elastic that demand at the point of initial equilibrium, owing to an increase in $U_{1.1}$ or a decrease in $|U_{1.11}|$, the smaller will be the volumes both of its exports and of its imports, and the more favorable to it will be the terms of trade. On the other hand, given an increase in the import-demand of country 1 resulting from an enlargement of its productive power, the more elastic that demand at the point of initial equilibrium, owing to a decrease in $|U_{1.22}|$, the larger will be the volumes both of its exports and of its imports, and the less favorable to it will be the terms of trade.

These conclusions relate to an increase in demand resulting from an enlargement of productive power. Consider now the case in which the increase in the import-demand of country 1 results from a change in its tastes, i.e., from a variation in its "utility" function. At a given condition of equilibrium $U_{1.2}$ will equal $\bar{p}_2U_{1.1}$. If now $U_{1.2}$ is modified by the addition of a constant quantity symbolized by $\delta U_{1.1}$, the change in tastes will favor X_2 and hence lead to an *increase* in import-demand if $[U_{1.2}+\delta U_{1.2}]/[U_{1.1}+\delta U_{1.1}]>\bar{p}_2$, or if $\delta U_{1.2}>\bar{p}_2\delta U_{1.1}$. Assume that the change in tastes is of this character.[14]

The foregoing change in tastes in country 1 will lead to a new set of equilibrium conditions. If it is assumed that both $\delta U_{1.2}$ and $\delta U_{1.1}$ are small, the resulting changes in the equilibrium values of the variables will be given approximately by a set of differential equations which will be identical with equations (4) except that the first of these equations will read $U_{1.22}d\overline{X}_{12}+\delta U_{1.2}=\bar{p}_2U_{1.11}d\overline{X}_{11}+U_{1.1}d\bar{p}_2+\bar{p}_2\delta U_{1.1}$ and the second will read $0=d\overline{X}_{11}+d\bar{x}_{11}$. When these differential equations are solved for $d\bar{x}_{12}$, $d\bar{x}_{11}$, and $d\bar{p}_2$, the solutions are found to be the same as equations (5) above except that $[\bar{p}_2\delta U_{1.1}-\delta U_{1.2}]$ replaces $\bar{p}_2U_{1.11}dX'_{11}$. Since $[\bar{p}_2\delta U_{1.1}-\delta U_{1.2}]$ is assumed to be negative and thus has the same sign as $\bar{p}_2U_{1.11}dX'_{11}$ when the change in underlying conditions is such as to effect an increase in demand, the new solutions for $d\bar{x}_{12}$, $d\bar{x}_{11}$, and $d\bar{p}_2$ have the same signs as before.

[14] Both $\delta U_{1.2}$ and $\delta U_{1.1}$ are taken to be constants in order to simplify the analysis. It is recognized that this is an assumption of a *particular* type of change in tastes.

It is also seen that in these new solutions $U_{1.1}$ and $U_{1.22}$ enter in the same manner as before, and therefore variations in elasticity arising from changes in these factors have the same effect as in the former case. The factor $U_{1.11}$, however, is now found only in the denominators of the solutions, and not in both numerators and denominators as previously. Hence variation in elasticity owing to a change in $U_{1.11}$ has just the opposite effect as formerly.[15] When the increase in the import-demand of country 1, therefore, results from an enhanced preference for imported commodities, the more elastic that demand at the initial point of equilibrium, owing to an increase in $U_{1.1}$, the smaller will be the volumes both of its imports and exports and the more favorable to it will be the terms of trade. On the other hand, given an increase in the import-demand of country 1 resulting from an enhancement of its preference for imported goods, the more elastic that demand at the initial point of equilibrium, owing to a decrease in $|U_{1.11}|$ or a decrease in $|U_{1.22}|$, the larger will be the volumes both of its imports and exports, and the less favorable to it will be the terms of trade. The change in the assumption as to the cause of the increase in demand thus changes the solution of the problem only in that case in which the variation in elasticity is ascribed to a decrease in $|U_{1.11}|$.

III. FACTORS DETERMINING THE DIRECTION OF SHIFT IN DEMAND

The foregoing algebraic analysis has shown that the specification of a particular change in underlying conditions will yield a particular solution of Marshall's problem. In the graphic analysis it was found that the direction in which the demand curve was shifted determined the solution of the problem. This suggests the existence of a definite relationship between certain changes in underlying conditions and shifts in demand in particular directions.

Care must be exercised, however, in making inferences which are based merely on the similarity in solutions. Figure 5 shows that positive shifts in demand may be represented graphically by projecting the demand curve in six fundamentally different directions. These are (I) shifts horizontally to the right and downward, (II) shifts horizontally to the right, (III) shifts horizontally to the right and upward, but at an angle that is less than the slope of the supply curve, (IV) shifts horizontally to the right and upward, but at an angle that is greater than the slope of the supply curve, (V) shifts vertically upward, and (VI) shifts vertically upward and to the left. If the reader will imagine

[15] Since $D=\bar{p}_2U_{1.11}A_{12}-\bar{p}_2U_{2.11}A_{22}+A_{32}$ is positive, a decrease in the absolute value of $U_{1.11}$ will decrease the value of D and hence increase the values of $|d\bar{x}_{12}|$, $d\bar{x}_{11}$, and $d\bar{p}_2$.

another demand curve passing through the point P of Figure 5 but less steep and hence more elastic at that point than the demand curve DD and if he will imagine it to be shifted in each of the above directions along with the curve DD, he will find that shifts in directions I, II, and III all yield similar solutions of Marshall's problem, while shifts in directions IV, V, and VI all yield solutions which are similar to each other but are different from those of the first group. This indicates the danger in associating certain changes in underlying conditions with shifts in particular directions merely because they yield similar solutions in the algebraic and graphic analyses respectively.

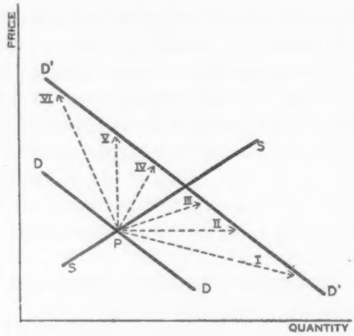

FIGURE 5

Fortunately, the analysis can be extended to give more definite conclusions. In Marshall's problem the direction of shift in demand is determined by the position of the point of intersection of the two demand curves after they have been shifted relative to the original position of this point, which is the same as the point of initial equilibrium. If simple formulas can be derived for these demand curves in terms of the given data, it should be possible to determine their points of intersection both before and after the shifting. Then, by noting how the relative position of these points is affected by a given change in underlying conditions, the direction of shift associated with that change may be determined. Let us pursue this method of analysis.

In the immediate neighborhood of the point of initial equilibrium, it may be assumed that the "final degree of utility" functions, $U_{1.1}$ and $U_{1.2}$, are approximately linear.[16] They will thus have the forms

[16] This is the same assumption as that made in the preceding analysis. Cf. footnote 8 above.

$U_{1.1}(X_{11}) = U_{1.1}(\overline{X}_{11}) + U_{1.11}(\overline{X}_{11})[X_{11} - \overline{X}_{11}]$ and $U_{1.2}(X_{12}) = U_{1.2}(\overline{X}_{12}) + U_{1.22}(\overline{X}_{12})[X_{12} - \overline{X}_{12}]$, where the $\overline{X}$'s represent equilibrium values. The first of equations (1) may then be written

$$U_{1.2}(\overline{X}_{12}) + U_{1.22}(\overline{X}_{12})[X_{12} - \overline{X}_{12}] = p_2U_{1.1}(\overline{X}_{11}) + p_2U_{1.11}(\overline{X}_{11})[X_{11} - \overline{X}_{11}].$$

When this is combined with the rest of equations (1), the import-demand function of country 1 is found to be $ap_2 + cp_2^2x_{12} + bx_{12} + k = 0$, where $a = U_{1.1}(\overline{X}_{11}) + U_{1.11}(\overline{X}_{11})[X'_{11} - \overline{X}_{11}]$, $c = U_{1.11}(\overline{X}_{11})$, $b = U_{1.22}(\overline{X}_{12})$, and $k = U_{1.22}(\overline{X}_{12})\overline{X}_{12} - U_{1.2}(\overline{X}_{12})$.

In the preceding section it was found that the elasticity of this demand curve at the initial point of equilibrium would be increased by an increase in $U_{1.1}$ or a decrease in either $|U_{1.22}|$ or $|U_{1.11}|$. If the increase in elasticity is obtained by increasing $U_{1.1}$ and $U_{1.2}$,[17] the new demand curve will have the form $a'p_2 + cp_2^2x_{12} + bx_{12} + k' = 0$, the c and b coefficients being the same for both the old and the new curves, since they are supposedly not affected by variation in $U_{1.1}$ and $U_{1.2}$. The p_2 co-ordinate of the intersection[18] of these two demand curves will thus be $\hat{p}_2 = (k' - k)/(a - a')$. If the shift in demand is caused by an increase in productive power, i.e., by an increase in X'_{11}, this will only change a and a' in the two demand functions. Furthermore, this variation in X'_{11} will cause variations in a and a' of equal amounts so that $a - a'$ will remain invariant. Hence, the p_2 co-ordinate of the point of intersection of the two demand curves after their shift will be the same as before the shift. Thus, with reference to graphs in which p_2 is measured vertically and x_{12} horizontally, it may be concluded that in this case the curves have shifted horizontally, i.e., in direction II of Figure 5. If the shift in demand is caused by a change in tastes, in particular by the addition of $\delta U_{1.1}$ to $U_{1.1}$ and of $\delta U_{1.2}$ to $U_{1.2}$, $\delta U_{1.1}$ and $\delta U_{1.2}$ being constants of such a size that $\delta U_{1.2} > \bar{p}_2\delta U_{1.1}$, then this will change the parameters a and a' and k and k' of the demand functions. Since it is assumed, however, that both curves undergo the same degree of shifting, i.e., the same variations in $U_{1.1}$ and $U_{1.2}$, it follows that $a - a'$ and $k' - k$ will not be affected by the shifting and the p_2 co-ordinate of the point of intersection will again remain invariant. Thus, whether the cause of the increase in demand is an increase in productive power or a change in tastes of the kind described, if the difference in elasticity

[17] Owing to the relationship $U_{1.2} = \bar{p}_2U_{1.1}$, these two quantities must vary together at the point of initial equilibrium.

[18] Co-ordinates of the point of intersection will be distinguished by a circumflex accent.

between the two demand curves is due to an increase in $U_{1.1}$, the shift in demand will be horizontal and to the right.[19]

If the difference in elasticity between the old and the new demand curves is due to a decrease in $|U_{1.22}|$, the new demand function will have the form $ap_2+cp_2^2x_{12}+b'x_{12}+k'=0$, the a and the c coefficients being the same for both the old and the new curves. The x_{12} co-ordinate of the point of intersection of the two demand curves will in this case be $\hat{x}_{12}=(k'-k)/(b-b')$. If the shift in demand is caused by an increase in X'_{11}, neither k, k', b, nor b' are affected, so the x_{12} co-ordinate of the point of intersection of the two demand curves remains invariant. If the shift in demand is caused by a change in tastes resulting from the addition of constant quantities to $U_{1.2}$ and $U_{1.1}$, both k and k' are affected, but since the variation in each is the same, $k'-k$ will not be changed and again the x_{12} co-ordinate of the point of intersection will remain invariant. Hence, whether the shift in demand results from an increase in productive power or a change in tastes of the kind described, if the difference in elasticity between the two curves is due to a decrease in $|U_{1.22}|$, the shift in demand will be vertically upward.[20]

[19] To assure ourselves that the shift is to the right, it need only be noted that since $d\hat{p}_2$ is zero,

$$d\hat{x}_{12} = \frac{-\bar{p}_2 da}{b + c\bar{p}_2^2}$$

when the cause of the shift in demand is a small increase in X'_{11}, and

$$d\hat{x}_{12} = \frac{-\bar{p}_2 da - dk}{b + c\bar{p}_2^2}$$

when the cause of the shift in demand is a change in tastes of the kind described. In the first case, $da=cdX'_{11}$ and is negative if dX'_{11} is positive, as is assumed. Hence $d\hat{x}_{12}$ is negative since both b and c are negative, and therefore $\hat{x}_{12}$, being itself negative, is absolutely increased. In the second case, $da=\delta U_{1.1}$ and $dk=-\delta U_{1.2}$, and since $\delta U_{1.2}>\bar{p}_2\delta U_{1.1}$ by assumption, it follows that the numerator of the formula for $d\hat{x}_{12}$ is positive. Hence, the denominator being negative as before, $d\hat{x}_{12}$ is again negative, and the absolute value of $\hat{x}_{12}$ is increased.

[20] Again we can assure ourselves that the shift is upward by noting that, since $d\hat{x}_{12}$ is zero,

$$d\hat{p}_2 = \frac{-\bar{p}_2 da}{a + 2c\bar{p}_2\bar{x}_{12}}$$

when the cause of the shift in demand is a small increase in X'_{11}, and

$$d\hat{p}_2 = \frac{-\bar{p}_2 da - dk}{a + 2c\bar{p}_2\bar{x}_{12}}$$

when the cause of the shift in demand is a change in tastes of the type described. As shown in the previous footnote, the numerators of these fractions are both positive, and a consideration of the definitions of the various quantities shows that the denominators are also positive. Hence $d\hat{p}_2$ is in both cases positive and $\hat{p}_2$ is increased.

If the difference in elasticity between the old and new demand curves is due to a decrease in $|U_{1.11}|$, the new demand curve will have the form $a'p_2+c'p_2^2x_{12}+bx_{12}+k=0$, the b and k coefficients remaining the same for both the old and the new curves. In this case there are three possible points of intersection of the old and the new curves, the p_2 co-ordinates of these points being $\hat{p}_2=0$ and

$$\hat{p}_2 = \frac{-k(c'-c) \pm \sqrt{k^2(c'-c)^2 - 4(ac'-a'c)(a-a')b}}{2(ac'-a'c)}.$$

The root $\hat{p}_2=0$ is ruled out as unrealistic. Both of the latter values of $\hat{p}_2$ are positive,[21] but an examination of the first and second derivatives of the demand functions shows that if both curves are to be elastic at the point of intersection, which is the only case being considered, then the larger of these two values is alone admissible.[22]

If there is an increase in demand arising from an increase in X'_{11}, the coefficients a and a' are affected. From the definition of the various coefficients, however, and on the assumption that X'_{11} varies the same amount in the case of both a and a', the quantity $ac'-a'c$ remains invariant. Hence the only variation in the above formula for the p_2

[21] From the definitions of the coefficients and from the assumption that the difference in elasticity is due to a decrease in $|U_{1.11}|$, it follows that $c'-c$ and $ac'-a'c$ are positive and $a-a'$ is negative. Hence, since $b=U_{1.22}(\bar{X}_{12})$ is negative, the square-root term is less than $-k(c'-c)$, which is positive, and therefore both roots of $\hat{p}_2$ are positive.

[22] The derivatives of the original demand function are

$$\frac{dx_{12}}{dp_2} = -\frac{a+2cp_2x_{12}}{b+cp_2^2} \text{ and } \frac{d^2x_{12}}{dp_2^2} = -\frac{2c\,[x_{12}+2p_2dx_{12}/dp_2]}{b+cp_2^2},$$

and similar formulas hold for the other demand curve with a' and c' in place of a and c. Since, on the assumption that $|U_{1.11}|$ decreases, $a'>a$, it follows that the slope of the old curve (i.e., the curve whose parameters are a and c) is greater at $p_2=0$ than the slope of the new. (Note that p_2 is assumed to be measured vertically so that the slope is dp_2/dx_{12}.) At the intermediate point of intersection the reverse must be true, while at the point of intersection for which p_2 has its highest value the relative value of the slopes will be the same as at $p_2=0$. This automatically rules out the intermediate point of intersection as a possible point of equilibrium, for it was shown in the previous section that under the assumption of elastic demand curves a decrease in $|U_{1.11}|$ at the initial point of equilibrium would yield a curve of greater slope. This reasoning is corroborated by an examination of the second derivatives which show that the original demand curve has the greater curvature at any point and that the point of inflection on each curve is the point for which the elasticity is $=-1/2$, while at lower values of p_2 the absolute value of the elasticity is less than $1/2$. Hence, if the original demand curve is to cross the new demand curve twice above $p_2=0$, it must make the first crossing at a point *below* its inflection point. This intermediate point of intersection therefore must be a point at which the demand is inelastic.

co-ordinate of the intersection point is a variation in the quantity $a-a'$. If the change in X'_{11} is small, then the variation in $\hat{p}_2$ is given approximately by

$$d\hat{p}_2 = \frac{-b}{\sqrt{R}}d(a - a') = \frac{-b(c - c')}{\sqrt{R}}\, dX'_{11},$$

where $R=k^2(c'-c)^2-4(ac'-a'c)(a-a')b$. On the assumption that $|U_{1.11}|$ decreases, $c-c'$ is negative. Hence, since R must be positive for real values of the variables and since b is negative, it follows that $d\hat{p}_2$ is negative if dX'_{11} is positive, as is assumed to be the case. The p_2 co-ordinate of the new point of intersection is therefore less than that of the old. The change in the x_{12} co-ordinate is given by

$$ad\hat{p}_2 + \bar{p}_2da + 2c\bar{p}_2\bar{x}_{12}d\hat{p}_2 + c\bar{p}^2d\hat{x}_{12} + bd\hat{x}_{12} = 0$$

or by

$$d\hat{x}_{12} = \frac{(a + 2c\bar{p}_2\bar{x}_{12})d\hat{p}_2 + \bar{p}_2da}{- b - c\bar{p}^2}\,.$$

The definitions of the various quantities show that $-b-c\bar{p}_2^2$ and $a+2c\bar{p}_2\bar{x}_{12}$ are positive, and it has just been seen that $d\hat{p}_2$ is negative. The quantity da equals cdX'_{11} and is therefore negative. Hence $d\hat{x}_{12}$ is negative. But $\hat{x}_{12}$ is negative, so that if $d\hat{x}_{12}$ is negative, it follows that the absolute value of $\hat{x}_{12}$ has increased. Therefore the x_{12} co-ordinate of new point of intersection is absolutely larger than that of the old point of intersection. Since $\hat{p}_2$ is thus absolutely smaller and $\hat{x}_{12}$ absolutely larger, it follows that the direction of shift has been horizontally to the right and downward, i.e., in direction I of Figure 5.

If there is an increase in demand arising from a change in tastes caused by the addition of $\delta U_{1.2}$ to $U_{1.2}$ and $\delta U_{1.1}$ to $U_{1.1.}$, $\delta U_{1.2}$ and $\delta U_{1.1}$ being constants of such a size that $\delta U_{1.2}>\bar{p}_2\delta U_{1.1}$, then the coefficients a and a' and k and k' are affected, the variation in a and a' being $\delta U_{1.1}$ and the variation in k and k' being $-\delta U_{1.2}$. If we restrict ourselves to a consideration of the special case in which $\delta U_{1.2}$ is positive and $\delta U_{1.1}$ is negative, the direction of shifting can be readily discovered. For if $\delta U_{1.2}$ is positive, then the variations in k and k' are negative and since both k and k' are themselves negative, the effect of the variation is to increase their absolute values. Similarly, if $\delta U_{1.1}$ is negative, the variations in a and a' will be such as to make the variation in $ac'-a'c$ negative.[23] Hence, on referring back to the above expression for the p_2 co-ordinate of the point of intersection and on noting

[23] Since $ac'-a'c=[c'-c]U_{1.1}$, the variation in $ac'-a'c$ equals $[c'-c]\delta U_{1.1}$. Since $|c'|$ is assumed to be less than $|c|$ and since both c and c' are negative, it follows that $c'-c$ is positive and $[c'-c]\delta U_{1.1}$ is negative if $\delta U_{1.1}$ is negative.

the signs of the various quantities, it will be found that the effect of the shift is to increase the value of the p_2 co-ordinate. Now if we turn back to the equations for the two demand functions, viz., $ap_2+cp_2^2x_{12}+bx_{12}+k=0$ and $a'p_2+c'p_2^2x_{12}+bx_{12}+k=0$ and subtract the first from the second, it will be found that the co-ordinates of the point of intersection must satisfy the condition $\hat{p}_2\hat{x}_{12}=(a-a')/(c'-c)$. But when the increase in demand is due to variation in $U_{1.2}$ and $U_{1.1}$ as described above, neither c nor c' are affected and a and a' are varied equally so that $a-a'$ remains invariant. Hence the new point of intersection as well as the old must satisfy the above condition. This means that if the p_2 co-ordinate of the point of intersection is increased by the shifting, as was seen to be the case, then the absolute value of the x_{12} co-ordinate must decrease. Since $\hat{p}_2$ is absolutely larger and $\hat{x}_{12}$ is absolutely smaller, it follows that the direction of shift has been upward and to the left, i.e., in direction VI of Figure V.

Thus, when the difference in elasticity between the old and new demand curves is due to a decrease in $|U_{1.11}|$ and when the increase in demand is due to an increase in productive power, the demand curves are shifted downward and to the right. On the other hand, when the difference in elasticity is due to a decrease in $|U_{1.11}|$, but the increase in demand is due to a change in tastes of the type described, the demand curves are shifted upward and to the left.[24]

IV. CONCLUSION

The foregoing results are, of course, dependent on the highly specialized set of assumptions laid down at the beginning of the analysis. They are peculiar to Marshall's problem. Nevertheless, they suggest certain general conclusions regarding the direction of shift of a demand curve. First, they show that a shift in demand may not always be a simple horizontal or vertical movement, but may take an intermediate direction or may even take a direction that would seem at first glance to be inconsistent with the positive or negative character of the shift, such as a *positive* shift *downward* and to the right, or a *positive* shift upward and to the *left*. Secondly, since the determination of the direction of shift apparently depends upon some means of identifying corresponding points on a curve before and after its shifting, it would seem that the direction of shift cannot be determined in those problems in which the data do not permit of such identification. It is likely that in these problems the direction of shift has no bearing on their solution.

[24] It will be noted that in this second case a more specialized type of change in tastes was assumed than in the other cases. Intuition suggests, however, that the conclusions are also valid for the broader case, but the writer has been unable to prove it.

On the other hand, where the direction of shift does have a bearing on the solution, it is likely that the problem will be such that corresponding points can be identified. Finally, the above results suggest that mere knowledge of the cause of the shift in demand is not likely to be sufficient to determine the direction of shift. In Marshall's problem it was found that for a given cause of the increase in demand, the particular way in which the elasticity of England's demand curve was varied determined the direction of shift. It was this difference in the manner of effecting the variation in elasticity that made it possible to identify the point of intersection of the two demand curves before and after the shift and hence to determine the direction of shift in general. It would seem that knowledge related in some way to the identification of corresponding points before and after the shift must be available before the direction of shift can itself be determined.[25]

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[25] Thus it would seem that such statements as "an increase in the number of buyers causes a demand curve to shift horizontally" and "an increase in the amount of purchasing power per consumer causes a demand curve to shift vertically" are not likely to be generally valid although they may be true for particular cases. Cf. G. Shepherd, "Vertical and Horizontal Shifts in Demand Curves," ECONOMETRICA, Vol. 4, p. 362.

THE THEORETICAL DERIVATION OF DYNAMIC DEMAND CURVES[1]

By Gerhard Tintner

It is the purpose of this paper to generalize the demand theory of Hicks and Allen[2] for the dynamic case. It also could give a somewhat firmer theoretical foundation to the dynamic demand theory of the Econometrists, especially G. C. Evans[3] and C. F. Roos.[4] We propose to derive income, price, and interest elasticities of demand under the assumption that the individual has definite plans for the future and definite expectations of future incomes, prices, and interest rates. Hence uncertainty in the sense of F. H. Knight[5] is ruled out, whereas risk may be taken into account. We make the same assumptions as in the previous paper on "Maximization of Utility over Time."[6] The individual plans for n discontinuous points in time in the discontinuous case, where utility is a mere function. Utility becomes a functional rather than a function in the continuous case.

A. THE DISCONTINUOUS CASE

We assume that the individual is at the point in time 0 and plans for the points $1, 2, \cdots, n$. He plans to consume m commodities $X_1, \cdots, X_m$. Let us denote the quantity of the commodity number v which he expects to consume at the point in time s by x_{vs}. Let p_{vs} be the expected price of this commodity at the point s. Denote by i_s the expected rate of interest at the point in time s and by $r_s = 1+i_s$ the corresponding accumulation factor for this period. We define by $R_s = r_1 r_2 \cdots r_s$ the total expected accumulation factor for the period $1, \cdots, s$. Denote by I_s the expected income at the period s. Then we define $J_s = I_s/R_s$ the discounted expected income for this period. We also define the discounted expected price of the commodity number v at the point in time s as $q_{vs} = p_{vs}/R_s$.

[1] Paper read at the Fourth Annual Research Conference of the Cowles Commission for Research in Economics, Colorado Springs, July, 1938. The author has to thank Mr. R. G. D. Allen (London School of Economics) very much for his kind help and especially for very valuable suggestions for the simplification of the notation of this paper.

[2] G. R. Hicks and R. G. D. Allen, "A Reconsideration of the Theory of Value," *Economica*, Vol. 1, 1934, pp. 52 ff. R. G. D. Allen: *Mathematical Analysis for Economists*, London, 1938, pp. 509 ff. G. R. Hicks: *Théorie Mathématique de la Valeur*, Paris, 1937.

[3] G. C. Evans: *Mathematical Introduction to Economics*, New York, 1930.

[4] C. F. Roos: *Dynamic Economics*, Bloomington, Ind., 1934.

[5] F. H. Knight: *Risk, Uncertainity and Profit.* Reprint, London, 1933.

[6] *Econometrica*, Vol. 6, 1938, pp. 154 ff.

We assume that the individual has an integrable[7] utility function F which depends upon all the quantities of all the commodities which he expects to consume over the period $1, \cdots, n$:

$$F = F(x_{11}, x_{12}, \cdots, x_{mn}). \tag{1}$$

This utility function may be replaced by an arbitrary utility index with positive derivative since only the ratios of the marginal utilities (marginal rates of substitution) enter into the theory.

The individual will maximize his utility function under the following condition (budget equation):

$$\sum_{s=1}^{n} \sum_{v=1}^{m} x_{vs} q_{vs} = \sum_{s=1}^{n} J_s = J, \tag{2}$$

where J is the total discounted income over the whole period $1, \cdots, n$. The meaning of equation (2) is that the total discounted expenditure must be equal to the total discounted income.[8]

We proceed now in the same manner as Hicks and Allen. We want, however, to give first a somewhat more general solution, from which income, price, and interest derivatives and elasticities of demand follow as special cases.

Differentiating (1) under the condition (2) and introducing a Lagrange multiplier λ (marginal utility of money) we get the following system:

$$-q_{vs}\lambda + F_{vs} = 0, \qquad (v = 1, \cdots, m; s = 1, \cdots, n), \tag{3}$$

where F_{vs} denotes the partial derivative of F with respect to x_{vs}. These mn equations (3) suffice together with the budget equation (2) to determine the values of the $mn+1$ quantities x_{vs} and λ, if certain further conditions are fulfilled. We get the demand of the commodity number v at the point in time s as a function of the expected discounted prices and incomes: $x_{vs} = x_{vs}(J_1, \cdots, J_n, q_{11}, \cdots, q_{mn})$. The dependence of these expected quantities upon past incomes, prices, and interest rates forms the bridge between our analysis and the dynamic demand functions of Evans and Roos. If this dependence is known then the demand appears as a function of past factors.

We are interested in the variation of the demand (x_{ut}) induced by variations of income (J, J_w, I_w), prices (q_{zw}, p_{zw}), as well as the accumulation rates (r_w). The most general solution can be obtained by the use of differentials.

[7] R. G. D. Allen, *op. cit.*, pp. 509 ff.

[8] G. Tintner, *loc. cit.*, p. 156.

Differentiating the budget equation (2) and the system (3) we get he following linear system:

$$\sum_{s=1}^{n}\sum_{v=1}^{m} q_{vs}dx_{vs} = -\sum_{s=1}^{n}\sum_{v=1}^{m} x_{vs}dq_{vs} + \sum_{s=1}^{n} dJ_s$$

$$(4) \qquad = -\sum_{s=1}^{n}\sum_{v=1}^{m} x_{vs}dq_{vs} + dJ$$

$$-q_{ut}d\lambda + \sum_{s=1}^{n}\sum_{v=1}^{m} F_{vs,ut}dx_{vs} = \lambda dq_{ut},$$

$$(u = 1, \cdots, m;\ t = 1, \cdots, n),$$

where $F_{vs,ut}$ is the second partial derivative of F with respect to x_{vs} and x_{ut}.

This system can be solved by determinants for $d\lambda$ and dx_{ut}. Let us introduce the determinant D:

$$(5) \qquad D = \begin{vmatrix} 0 & F_{11} & F_{12} & \cdots F_{mn} \\ F_{11} & F_{11,11} & F_{11,12} & \cdots F_{11,mn} \\ \cdot & \cdot & \cdot & \cdot \\ F_{mn} & F_{mn,11} & F_{mn,12} & \cdots F_{mn,mn} \end{vmatrix}.$$

Denote by D_{vs} the minor of F_{vs} and by $D_{vs,ut}$ the minor of $F_{vs,ut}$.

The solution of the linear system (4) appears in the following form, if we substitute from (3):

$$(6) \qquad dx_{ut} = \frac{\lambda}{D}\left[\left(-\sum_{s=1}^{n}\sum_{v=1}^{m} x_{vs}dq_{vs} + \sum_{s=1}^{n} dJ_s\right)D_{ut} + \sum_{s=1}^{n}\sum_{v=1}^{m} dq_{vs}D_{vs,ut}\right].$$

This is the most general solution for the demand differential dx_{ut}. There may be certain connections between the expected values of the I, p, and r, but this solution would take care of them. We will however restrict ourselves to the usual approach and vary only one quantity at a time, the others being held constant.

Specializations of formula (6) give us the following results:

$$(7) \qquad \frac{\partial x_{ut}}{\partial J_w} = \frac{\lambda D_{ut}}{D}, \text{ (income derivative)},$$

$$(8) \qquad \frac{\partial x_{ut}}{\partial q_{zw}} = \frac{\lambda(-x_{zw}D_{ut} + D_{ut,zw})}{D}, \text{ (price derivative)}.$$

According to its definition $\partial J_s/\partial r_w = 0$ for $s < w$ and $= -J_s/r_w$ for $s \geqq w$; and $\partial q_{vs}/\partial r_w = 0$ for $s < w$ and $= -q_{vs}/r_w$ for $s \geqq w$. Hence we get the following interest-rate (or rather accumulation-rate) derivatives:

$$(9)\qquad \frac{\partial x_{ut}}{\partial r_w} = \frac{-\lambda}{r_w D}\left\{\left[\sum_{s=w}^{n}\left(J_s - \sum_{v=1}^{m} x_{vs}q_{vs}\right)\right]D_{ut} + \sum_{s=w}^{n}\sum_{v=1}^{m} q_{vs}D_{vs,ut}\right\}.$$

It should be noted that the sum in square brackets is the expected discounted saving in the period $w, \cdots, n$, i.e., the expected discounted income minus the expected discounted expenditure of this period, discounted with the expected rates of interest.

Following again the procedure of Hicks and Allen we want to express some interesting relationships between elasticities. There are the following relationships between the elasticities of demand with respect to total discounted income (J), discounted (J_w), and undiscounted income at the point w (I_w):

$$(10)\qquad \frac{Ex_{ut}}{EJ_w} = \frac{Ex_{ut}}{EI_w} = k_w \frac{Ex_{ut}}{EJ},$$

where $k_w = J_w/J$ is the proportion of the discounted income in the period w to the total discounted income. The first two income elasticities in formula (10) are equal because J_w is a constant multiple of I_w.[9] The elasticity of demand with respect to the discounted or undiscounted income expected for the point in time w is equal to the elasticity of demand with respect to the total discounted income multiplied by the proportion of the discounted income at the point in time w to the total discounted income.

In order to state the price elasticity of demand in a similar manner as in the static case we have to define two quantities: Let $k_{zw} = (x_{zw}q_{zw})/J$ be the proportion of the total discounted income spent on the commodity number z at the point in time w. We define further an elasticity of substitutions:

$$(11)\qquad \sigma_{ut,zw} = \frac{D_{ut,zw}\sum_{s=1}^{n}\sum_{v=1}^{m} x_{vs}F_{vs}}{x_{ut}x_{zw}D}.$$

The elasticity of demand for the commodity number u at the point t with respect to the discounted or undiscounted price for the commodity number z at the point in time w appears as follows:

$$(12)\qquad \begin{aligned} \frac{Ex_{ut}}{Eq_{zw}} = \frac{Ex_{ut}}{Ep_{zw}} &= -k_{zw}\frac{Ex_{ut}}{EJ} + k_{zw}\sigma_{ut,zw} \\ &= -\frac{k_{zw}}{k_w}\frac{Ex_{ut}}{EJ_w} + k_{zw}\sigma_{ut,zw} \\ &= -\frac{k_{zw}}{k_w}\frac{Ex_{ut}}{EI_w} + k_{zw}\sigma_{ut,zw}. \end{aligned}$$

[9] R. G. D. Allen, *op. cit.*, p. 253.

This is very similar to the formula for the static case.[10] The elasticity of demand with respect to the total discounted income is multiplied by the proportion of the discounted income spent on the commodity at the point in time in question. The multiplier for the elasticity of demand with respect to the discounted or undiscounted income at the point in time w is the ratio of the relative proportion spent on the commodity at the point in time to the total discounted income divided by the relative proportion of the expected income in w to the total discounted income. This is the same as the ratio of the expected expenditure on the commodity at the point in time in question to the expected income at this point in time.

The elasticities of demand with respect to the expected interest or rather accumulation rate have not been derived before. Starting from equation (9) and making use of the results already derived we get:

$$
\begin{aligned}
\frac{Ex_{ut}}{Er_w} &= -\frac{Ex_{ut}}{EJ}\,\frac{\sum_{s=w}^{n}\left(J_s-\sum_{v=1}^{m}x_{vs}q_{vs}\right)}{J}-\sum_{s=w}^{n}\sum_{v=1}^{m}k_{vs}\sigma_{ut,vs}\\
&= -\frac{Ex_{ut}}{EJ}\sum_{s=w}^{n}\left(k_s-\sum_{v=1}^{m}k_{vs}\right)-\sum_{s=w}^{n}\sum_{v=1}^{m}k_{vs}\sigma_{ut,vs}\\
&= -\frac{Ex_{ut}}{EJ_w}\,\frac{\sum_{s=w}^{n}\left(J_s-\sum_{v=1}^{m}x_{vs}q_{vs}\right)}{J_w}-\sum_{s=w}^{n}\sum_{v=1}^{m}k_{vs}\sigma_{ut,vs}.
\end{aligned}
\tag{13}
$$

The coefficient of the income elasticity is always the discounted saving expected for the period $w, \cdots, n$, i.e., the difference between the expected income and the expected expenditure for this period. The second part of the formula is the sum of elasticities of substitution for the same period multiplied by the ratios of the expenditures to the total discounted income.

Making use of the property that by definition [from (2)] $\sum_{s=1}^{n}\sum_{v=1}^{m}x_{vs}q_{vs}=J$ and as in the static case $\sum_{s=1}^{n}\sum_{v=1}^{m}k_{vs}\sigma_{ut,vs}=0$, we can change the limits of summation:[11]

$$
\frac{Ex_{ut}}{Er_w}=\frac{Ex_{ut}}{EJ}\,\frac{\sum_{s=1}^{w-1}\left(J_s-\sum_{v=1}^{m}x_{vs}q_{vs}\right)}{J}+\sum_{s=1}^{w-1}\sum_{v=1}^{m}k_{vs}\sigma_{ut,vs}.
\tag{14}
$$

The elasticity of demand with respect to income is here multiplied by

[10] G. R. Hicks, *op. cit.*, p. 14.

[11] This formula is due to Mr. R. G. D. Allen.

the expected saving in the period $1, \cdots, w-1$. The elasticities of substitution are summed over the same period.

B. THE CONTINUOUS CASE

Utility appears here as a functional and not as a mere function. The utility functional f is fixed if all the quantities are fixed, which the individual anticipates consuming in the time interval $0, \cdots, n$:

$$f = f\left[\overset{n}{\underset{0}{x_1(t)}},\ \overset{n}{\underset{0}{x_2(t)}},\ \cdots,\ \overset{n}{\underset{0}{x_m(t)}}\right]. \tag{15}$$

This functional is to be maximized under the following condition:

$$\sum_{v=1}^{m}\int_0^n x_v(s)q_v(s)ds = \int_0^n J(s)ds = J. \tag{16}$$

The maximum conditions appear here as:[12]

$$f_v'[x_1, \cdots, x_m; s] = \Lambda q_v(s), \qquad (v = 1, \cdots, m; 0 \leqq s \leqq n). \tag{17}$$

The first expression in this formula is the functional derivative with respect to x_v at the point in time s. Λ is a Lagrange multiplier (functional marginal utility of money) to be determined by (16).

Starting from the budget equation (16) and the system (17) we get the following system of integral equations for the determination of the functional elasticities of demand with respect to y[y may be any of the quantities J, $J(w)$, $I(w)$, $q_z(w)$, $p_z(w)$, $r(w)$]:

$$\begin{aligned} \sum_{v=1}^{m}\int_0^n x_v(s)q_v(s)\,\frac{Ex_v(s)}{Ey}\,ds &= -\sum_{v=1}^{m}\int_0^n x_v(s)q_v(s)\,\frac{Eq_v(s)}{Ey}\,ds \\ &\quad + \int_0^n J(s)\,\frac{EJ(s)}{Ey}\,ds, \\ \sum_{v=1}^{m}\int_0^n x_v(s)f_{vu}''[x_1, \cdots, x_m; s, t]\,\frac{Ex_v(s)}{Ey}\,ds & \\ &= \Lambda q_u(t)\left[\frac{E\Lambda}{Ey} + \frac{Eq_u(t)}{Ey}\right], \\ &\qquad (u = 1, \cdots, m; 0 \leqq t \leqq n). \end{aligned} \tag{18}$$

The income, price, and interest elasticities can easily be derived from this general system of equations.

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[12] H. Hahn, "Ueber die Lagrangesche Multiplikatorenmethode," *Sitzungsberichte der Akademie Wien*, Vol. 131, 1922, pp. 531 ff. H. H. Goldstine, "The Minima of Functionals with Associated Side Conditions," *Duke Mathematical Journal*, Vol. 3, 1937, pp. 418 ff.

A MATHEMATICAL NOTE ON DEMOCRACY

By K.-G. Hagstroem

Let $j(x)$ be the fraction of the annual income x which is taken by the State as yearly tax, $k(X)$ the yearly income which the State takes out of a capital X per unit of capital, and $h(Y)$ the fraction of an inheritance Y which is taken by the State. By determining the functions $j(x)$, $k(X)$, and $h(Y)$ on a high level, the State can approach communism at pleasure, always clinging formally to the principle of the right of private property. An interesting mathematical problem is:

Is it possible to give conditions which these functions have to fulfil in order to maintain some kind of equity in determining the taxes?

We shall not attempt a systematic discussion of this far-reaching problem but only suggest an idea which connects the problem with the income distribution.

Let x_0 be the value of the minimum consumption (the minimum of existence) of an individual, and suppose that out of the population a certain proportion is supported (children, old or sick people, unemployed), the fraction $(1-\omega)$ of the population being the real workers, whose productive labour will create the national income. For simplicity, we shall put the income of the supported $=0$. If n is the number of inhabitants, we have consequently ωn supported persons and $(1-\omega)n$ workers. Suppose further, that the income of the *workers* is distributed according to the famous law of Pareto. If the probability of an income of x units of money or more is proportional to the function x^{-k}, we easily conclude that the factor of proportionality must be such that the probability in question is:

$$\sigma(x) = (1-\omega)\left(\frac{x}{x_0}\right)^{-k}, \qquad (\text{for } x \geqq x_0).$$

The constant k will be about 1.3, in some extreme cases rising as high as 1.9, whereas its lowest practical value seems to be about 1.2. We shall make another fundamental hypothesis, viz., that the function $\sigma(x)$ is independent of the numbers ω and x_0.

The total yearly income of the population will be found by multiplying the number n by the mean value of the income

$$-\int_{x_0}^{\infty} x d\sigma(x) = -\int_{x_0}^{\infty} x d\sigma(x) = \frac{k}{k-1}(1-\omega)x_0,$$

or approximately $=4\frac{1}{3}x_0(1-\omega)$. It is a general property of the function σ that the mean income over x per individual having incomes over x is equal to $xk/(k-1)$.

Putting the total income equal to nx_0, we obtain

$$\frac{k}{k-1}(1-\omega)nx_0 = nx_0$$

or

$$\omega = \frac{1}{k}, \quad \text{approximately} = 77\%.$$

This gives an interesting relation between the constant k and the proportion of supported persons.

Let $J(x)=xj(x)$ be the amount of the tax. The total tax (per individual of the population) will be $\int_0^\infty J(x)d\sigma(x)$. A tax proportional to the amount of income above the minimum of existence is defined by

$$J(x) = 0 \quad \text{for} \quad x \leqq x_0,$$
$$J(x) = \epsilon\,(x - x_0) \quad \text{for} \quad x \geqq x_0.$$

In this case we obtain for the total tax

$$-\int_{x_0}^{\infty} J(x)d\sigma(x) = \int_{x_0}^{\infty} \sigma(x)J'(x)dx = \frac{\epsilon(1-\omega)}{k-1}x_0.$$

If we distribute this tax, calculated for the whole population, between the ωn supported individuals, we get the relation

$$\frac{\epsilon(1-\omega)x_0 n}{k-1} = \omega x_0 n,$$

or, for a given ω,

$$\epsilon = (k-1)\frac{\omega}{1-\omega}.$$

The limit point, where majority is just exactly obtained by the Supported Party, will correspond to $\omega=\frac{1}{2}$, and in that case we have

$$\epsilon = k - 1, \text{ approximately} = 30\%.$$

The tax percentage will in this case be "only" 30%, and it is seen that many countries are in a certain neighborhood of this value.

For the "qualified" majorities of $\frac{2}{3}$ or $\frac{3}{4}$ we obtain $\epsilon=60\%$ or 90%, respectively. The difficulty can only be removed by introducing in the Magna Charta of the country a proviso that *a tax exceeding the fraction ϵ of the excess of a person's income above the minimum of consumption*

can not be imposed unless the deciding majority exceeds the fraction $\frac{\epsilon}{k-1+\epsilon}+\eta$, *where* η *is a positive number.*

But this stipulation is certainly not inherent in the pure idea of democracy. It need not be said that there are other obstacles to prevent a majority of supported persons from introducing excessive taxation. But the simple model discussed may nevertheless be suggestive and interesting.

Stockholm, Sweden

ANNOUNCEMENT OF THE DETROIT MEETING, DECEMBER 27–30, 1938

The December meeting of the Econometric Society will be held in conjunction with those of the other social science associations at Detroit, Michigan, Tuesday through Friday, December 27–30, 1938. A joint session with the American Statistical Association is planned for Tuesday evening, December 27. The Book-Cadillac Hotel will be the Society's headquarters during the meetings.

Papers on a variety of econometric topics will be presented by J. S. Bain, Harvard University; John Maurice Clark, Columbia University; Harold T. Davis, Northwestern University and Cowles Commission for Research in Economics; Paul H. Douglas, University of Chicago; Irving Fisher, Yale University; Elizabeth W. Gilboy, Harvard University; Harold Hotelling, Columbia University; Wassily Leontief, Harvard University; A. P. Lerner, London School of Economics; Alfred J. Lotka, Metropolitan Life Insurance Co.; Francis McIntyre, Cowles Commission and Colorado College; Karl Pribram, Social Security Board; Charles F. Roos, Institute of Applied Econometrics, Inc.; W. A. Shewhart, Bell Telephone Laboratories; A. Wald, Cowles Commission; and other speakers.

Any inquiries regarding these sessions should be addressed to Francis McIntyre, Secretary of Program Committee, Econometric Society, Colorado Springs.

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As of October 1, 1938.

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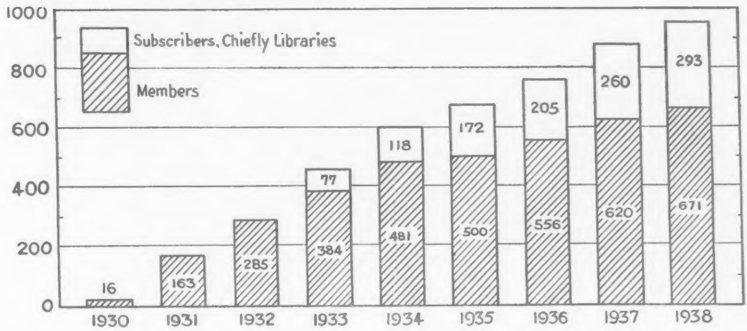
SUMMARY

Members, October 1, 1937		620
New Members Added		81
		701
Less: Deaths	5	
Resignations	21	
Dropped for nonpayment of dues	4	30
Members, October 1, 1938		671

GROWTH AND DISTRIBUTION OF MEMBERSHIP IN THE ECONOMETRIC SOCIETY AND OF SUBSCRIBERS TO ECONOMETRICA



GEOGRAPHIC DISTRIBUTION OF MEMBERS AND SUBSCRIBERS

	Members	Subscribers	Total		Members	Subscribers	Total
United States	238	116	354	Finland	3	2	5
England	53	20	73	Scotland	3	2	5
Italy	45	24	69	Yugoslavia	4	1	5
France	55	10	65	Chile	2	2	4
Netherlands	32	5	37	Brazil	1	2	3
Germany	28	11	39	Spain	3	....	3
Japan	13	18	31	Dutch E. Indies	1	1	2
Poland	17	9	26	Greece	....	2	2
Norway	21	3	24	Mexico	1	1	2
Switzerland	20	2	22	Algiers	1	....	1
Hungary	19	2	21	Cuba	....	1	1
Sweden	14	5	19	El Salvador	1	....	1
Belgium	15	3	18	Estonia	1	....	1
China	11	7	18	Latvia	....	1	1
Denmark	14	1	15	Lithuania	....	1	1
India	8	7	15	Philippines	1	....	1
Czechoslovakia	13	1	14	Puerto Rico	....	1	1
Argentina	5	5	10	Straits S.	....	1	1
Australia	4	6	10	Tasmania	....	1	1
Egypt	5	3	8	Turkey	1	....	1
Rumania	6	1	7	Uruguay	....	1	1
U. S. S. R.	....	7	7	Wales	....	1	1
Bulgaria	5	1	6				
Canada	3	3	6	Total	671	293	964
South Africa	4	2	6				

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